

## TWO PARTITION FUNCTIONS WITH CONGRUENCES MODULO 3, 5, 7, AND 13

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ABSTRACT. We introduce two new integer partition functions, both of which are the number of partition quadruples of  $n$  with certain size restrictions. We prove both functions satisfy Ramanujan-type congruences modulo 3, 5, 7, and 13 by use of generalized Lambert series identities and  $q$ -series techniques.

## 1. INTRODUCTION

We recall a partition of a positive integer  $n$  is a non-increasing sequence of positive integers that sum to  $n$ . For example, the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. We let  $p(n)$  denote the number of partitions of  $n$ . The function  $p(n)$  satisfies the well known congruences of Ramanujan  $p(5n + 4) \equiv 0 \pmod{5}$ ,  $p(7n + 5) \equiv 0 \pmod{7}$ , and  $p(11n + 6) \equiv 0 \pmod{11}$ . In this article we will consider two partition quadruples of  $n$ . We say a quadruple  $(\pi_1, \pi_2, \pi_3, \pi_4)$  of partitions is a partition quadruple of  $n$  is altogether the parts of  $\pi_1, \pi_2, \pi_3$ , and  $\pi_4$  sum to  $n$ .

For a partition  $\pi$ , we let  $s(\pi)$  denote the smallest part of  $\pi$  and  $\ell(\pi)$  denote the largest part of  $\pi$ . We use the conventions that the empty partition has smallest part  $\infty$  and largest part 0. We let  $u(n)$  denote the number of partition quadruples  $(\pi_1, \pi_2, \pi_3, \pi_4)$  of  $n$  such that  $\pi_1$  is non-empty,  $s(\pi_1) \leq s(\pi_2)$ ,  $s(\pi_1) \leq s(\pi_3)$ ,  $s(\pi_1) \leq s(\pi_4)$ , and  $\ell(\pi_4) \leq 2s(\pi_1)$ . We let  $v(n)$  denote the number of partition quadruples  $(\pi_1, \pi_2, \pi_3, \pi_4)$  of  $n$  such that the smallest part of  $\pi_1$  appears at least twice,  $s(\pi_1) \leq s(\pi_2)$ ,  $s(\pi_1) \leq s(\pi_3)$ ,  $s(\pi_1) \leq s(\pi_4)$ , and  $\ell(\pi_4) \leq 2s(\pi_1)$ .

We use the standard product notation,

$$\begin{aligned} (z; q)_n &= \prod_{j=0}^{n-1} (1 - zq^j), & (z; q)_\infty &= \prod_{j=0}^{\infty} (1 - zq^j), \\ (z_1, \dots, z_k; q)_n &= (z_1; q)_n \cdots (z_k; q)_n, & (z_1, \dots, z_k; q)_\infty &= (z_1; q)_\infty \cdots (z_k; q)_\infty, \\ [z; q]_\infty &= (z, q/z; q)_\infty, & [z_1, \dots, z_k; q]_\infty &= [z_1; q]_\infty \cdots [z_k; q]_\infty. \end{aligned}$$

By summing according to  $n$  being the smallest part of a partition, one easily deduces that a generating function for  $p(n)$  is given by

$$F(q) = \sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q^n; q)_\infty}.$$

Similarly, by summing according to  $n$  being the smallest part of the partition  $\pi_1$ , we find that generating functions for  $u(n)$  and  $v(n)$  are given by

$$\begin{aligned} U(q) &= \sum_{n=0}^{\infty} u(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(q^n; q)_\infty (q^n; q)_\infty (q^n; q)_\infty (q^n; q)_{n+1}}, \\ V(q) &= \sum_{n=0}^{\infty} v(n)q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{(q^n; q)_\infty (q^n; q)_\infty (q^n; q)_\infty (q^n; q)_{n+1}}. \end{aligned}$$

The main result of this article are the following congruences for  $u(n)$  and  $v(n)$ .

**Theorem 1.1.**

$$\begin{aligned} u(3n) &\equiv 0 \pmod{3}, \\ u(5n) &\equiv 0 \pmod{5}, \end{aligned}$$

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$$\begin{aligned}
u(5n+3) &\equiv 0 \pmod{5}, \\
u(7n) &\equiv 0 \pmod{7}, \\
u(7n+5) &\equiv 0 \pmod{7}, \\
u(13n) &\equiv 0 \pmod{13}, \\
v(3n+1) &\equiv 0 \pmod{3}, \\
v(5n+1) &\equiv 0 \pmod{5}, \\
v(5n+4) &\equiv 0 \pmod{5}, \\
v(13n+10) &\equiv 0 \pmod{13}.
\end{aligned}$$

To prove Theorem 1.1 we use  $q$ -series techniques and identities between generalized Lambert series to completely determine  $U(q)$  and  $V(q)$  modulo  $\ell$  for  $\ell = 3, 5, 7$ , and  $13$ . These results are stated in the following Theorem. For brevity we use the notation  $E(a) = (q^a; q^a)_\infty$  and  $P(a) = [q^{\ell a}; q^{\ell^2}]_\infty$ . That is to say, in the modulo 3 congruences  $P(a) = [q^{3a}; q^9]_\infty$ , in the modulo 5 congruences  $P(a) = [q^{5a}; q^{25}]_\infty$ , in the modulo 7 congruences  $P(a) = [q^{7a}; q^{49}]_\infty$ , and in the modulo 13 congruences  $P(a) = [q^{13a}; q^{169}]_\infty$ . We format the modulo 13 congruence differently from the other congruences due to the large number of terms.

**Theorem 1.2.**

$$U(q) \equiv \frac{qE(9)^2}{E(3)P(1)} + \frac{2q^2}{E(9)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{9n^2+15n}{2}}}{1-q^{9n+3}} \pmod{3}, \quad (1.1)$$

$$V(q) \equiv \frac{2q^3}{E(9)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{9n^2+15n}{2}}}{1-q^{9n+6}} + \frac{q^2}{E(9)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{9n^2+15n}{2}}}{1-q^{9n+3}} \pmod{3}, \quad (1.2)$$

$$\begin{aligned}
U(q) &\equiv \frac{qE(25)^2P(2)}{E(5)P(1)} + \frac{4q^2}{E(25)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{25n^2+35n}{2}}}{(1-q^{25n+5})} + \frac{q^2E(25)^2}{E(5)} + \frac{4q^4}{E(25)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{25n^2+35n}{2}}}{(1-q^{25n+10})} \\
&\pmod{5},
\end{aligned} \quad (1.3)$$

$$\begin{aligned}
V(q) &\equiv \frac{4q^5}{E(25)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{25n^2+35n}{2}}}{(1-q^{25n+15})} + \frac{q^2}{E(25)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{25n^2+35n}{2}}}{(1-q^{25n+5})} + \frac{4q^3E(25)^2P(1)}{E(5)P(2)} \\
&\pmod{5},
\end{aligned} \quad (1.4)$$

$$\begin{aligned}
U(q) &\equiv \frac{5q}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+7}} + \frac{3qE(49)^4P(2)^2P(3)}{E(7)P(1)^3} + \frac{4q^8E(49)^4P(2)^3}{E(7)P(1)P(3)^2} + \frac{3q^8E(49)^4P(1)P(3)^2}{E(7)P(2)^3} \\
&+ \frac{4q^2E(49)^4P(3)^2}{E(7)P(1)^2} + \frac{q^2E(49)^4P(2)^3}{E(7)P(1)^3} + \frac{q^9E(49)^4P(1)P(3)}{E(7)P(2)^2} + \frac{2q^9E(49)^4P(2)}{E(7)P(3)} + \frac{3q^3E(49)^4P(2)P(3)}{E(7)P(1)^2} \\
&+ \frac{4q^3E(49)^4P(2)^4}{E(7)P(1)^3P(3)} + \frac{q^3E(49)^4P(3)^3}{E(7)P(2)^2P(1)} + \frac{4q^{10}E(49)^4P(1)}{E(7)P(2)} + \frac{5q^{10}E(49)^4P(2)^2}{E(7)P(3)^2} \\
&+ \frac{2q^4}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+14}} + \frac{4q^6}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+21}} + \frac{2q^6E(49)^4P(3)^2}{E(7)P(2)^2} \\
&+ \frac{q^6E(49)^4P(2)}{E(7)P(1)} + \frac{4q^{13}E(49)^4P(1)^2}{E(7)P(2)P(3)} + \frac{6q^{13}E(49)^4P(1)P(2)^2}{E(7)P(3)^3} \pmod{7},
\end{aligned} \quad (1.5)$$

$$\begin{aligned}
V(q) &\equiv \frac{6q^7}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+28}} + \frac{2q}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+7}} + \frac{5qE(49)^4P(2)^2P(3)}{E(7)P(1)^3} \\
&+ \frac{3q^8E(49)^4}{E(7)} + \frac{6q^8E(49)^4P(2)^3}{E(7)P(1)P(3)^2} + \frac{q^{15}E(49)^4P(1)^3}{E(7)P(3)P(2)^2} + \frac{q^2E(49)^4P(2)^3}{E(7)P(1)^3} + \frac{3q^9E(49)^4P(3)P(1)}{E(7)P(2)^2} \\
&+ \frac{q^9E(49)^4P(2)}{E(7)P(3)} + \frac{4q^3E(49)^4P(2)^4}{E(7)P(1)^3P(3)} + \frac{5q^{10}E(49)^4P(1)}{E(7)P(2)} + \frac{4q^{10}E(49)^4P(2)^2}{E(7)P(3)^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q^4}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+14}} + \frac{4q^5 E(49)^4 P(3)}{E(7)P(1)} + \frac{5q^{12} E(49)^4 P(1)^2}{E(7)P(2)^2} + \frac{q^{12} E(49)^4 P(1)P(2)}{E(7)P(3)^2} \\
& + \frac{5q^6}{E(49)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{49n^2+63n}{2}}}{1-q^{49n+21}} + \frac{2q^6 E(49)^4 P(2)}{E(7)P(1)} + \frac{6q^{13} E(49)^4 P(1)^2}{E(7)P(2)P(3)} + \frac{4q^{13} E(49)^4 P(1)P(2)^2}{E(7)P(3)^3} \\
& \pmod{7},
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
U(q) & \equiv \frac{12q^3}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+39}} + \frac{10q^{-8}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+13}} \\
& + \frac{11q^7}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+52}} + \frac{q^{10}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+65}} \\
& + \frac{8q^{-2}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+26}} + \frac{4q^{12}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+78}} + A_{13}(q) \pmod{13},
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
V(q) & \equiv \frac{12q^{13}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+91}} + \frac{11q^3}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+39}} \\
& + \frac{3}{q^8 E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+13}} + \frac{12q^7}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+52}} \\
& + \frac{9}{q^2 E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+26}} + \frac{5q^{12}}{E(169)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{169n^2+195n}{2}}}{1-q^{169n+78}} \\
& + B_{13}(q) \pmod{13},
\end{aligned} \tag{1.8}$$

where

$$\begin{aligned}
A_{13}(q) & = \frac{E(169)^4}{E(13)} \left( A_{13,0}(q^{13}) + qA_{13,1}(q^{13}) + q^2A_{13,2}(q^{13}) + q^3A_{13,3}(q^{13}) + q^4A_{13,4}(q^{13}) \right. \\
& + q^5A_{13,5}(q^{13}) + q^6A_{13,6}(q^{13}) + q^7A_{13,7}(q^{13}) + q^8A_{13,8}(q^{13}) + q^9A_{13,9}(q^{13}) + q^{10}A_{13,10}(q^{13}) \\
& \left. + q^{11}A_{13,11}(q^{13}) + q^{12}A_{13,12}(q^{13}) \right),
\end{aligned}$$

$$\begin{aligned}
B_{13}(q) & = \frac{E(169)^4}{E(13)} \left( B_{13,0}(q^{13}) + qB_{13,1}(q^{13}) + q^2B_{13,2}(q^{13}) + q^3B_{13,3}(q^{13}) + q^4B_{13,4}(q^{13}) \right. \\
& + q^5B_{13,5}(q^{13}) + q^6B_{13,6}(q^{13}) + q^7B_{13,7}(q^{13}) + q^8B_{13,8}(q^{13}) + q^9B_{13,9}(q^{13}) + q^{10}B_{13,10}(q^{13}) \\
& \left. + q^{11}B_{13,11}(q^{13}) + q^{12}B_{13,12}(q^{13}) \right),
\end{aligned}$$

$$A_{13,0}(q^{13}) = 0,$$

$$\begin{aligned}
A_{13,1}(q^{13}) & = \frac{P(3)^3 P(4)^6}{P(1)^2 P(5)^2 P(2) P(6)} + \frac{9q^{13} P(3)^5 P(4)^3}{P(1)^2 P(5)^2 P(6)} + \frac{10q^{26} P(3)^{10}}{P(1) P(5)^2 P(2)^2 P(6) P(4)} + \frac{6q^{26} P(3)^7 P(2)}{P(1)^2 P(5)^2 P(6)} + \frac{6q^{39} P(3)^2 P(2)^5 P(4)}{P(1)^2 P(5)^2 P(6)} \\
& + \frac{7q^{39} P(3)^5 P(2)^2}{P(1) P(5)^2 P(6)} + \frac{7q^{52} P(3)^3 P(2)^3}{P(5)^2 P(6)} + \frac{3q^{52} P(1) P(3)^6}{P(5)^2 P(6) P(4)} + \frac{9q^{65} P(1) P(3) P(2)^4}{P(5)^2 P(6)} + \frac{12q^{65} P(1)^2 P(3)^4 P(2)}{P(5)^2 P(6) P(4)} \\
& + \frac{9q^{78} P(1)^3 P(3)^2 P(2)^2}{P(5)^2 P(6) P(4)} + \frac{10q^{91} P(1)^6 P(4)}{P(5)^2 P(3) P(6)} + \frac{9q^{91} P(1)^4 P(2)^3}{P(5)^2 P(6) P(4)} + \frac{3q^{104} P(1)^8}{P(5)^2 P(2)^2 P(6)}, \\
A_{13,2}(q^{13}) & = \frac{5P(3)^3 P(4)^6}{P(1)^2 P(5) P(2) P(6)^2} + \frac{10q^{13} P(3)^5 P(4)^3}{P(1)^2 P(5) P(6)^2} + \frac{8q^{26} P(3)^6 P(4)^2}{P(5) P(2)^2 P(6)^2} + \frac{q^{26} P(3)^7 P(2)}{P(1)^2 P(5) P(6)^2} + \frac{10q^{26} P(3)^{10}}{P(1) P(5) P(4) P(2)^2 P(6)^2} \\
& + \frac{7q^{39} P(3)^2 P(4) P(2)^5}{P(1)^2 P(5) P(6)^2} + \frac{11q^{39} P(3)^5 P(2)^2}{P(1) P(5) P(6)^2} + \frac{6q^{52} P(4) P(2)^6}{P(1) P(5) P(6)^2} + \frac{9q^{52} P(3)^3 P(2)^3}{P(5) P(6)^2} + \frac{8q^{52} P(1) P(3)^6}{P(5) P(4) P(6)^2} \\
& + \frac{3q^{65} P(1) P(3) P(2)^4}{P(5) P(6)^2} + \frac{11q^{65} P(1)^2 P(3)^4 P(2)}{P(5) P(4) P(6)^2} + \frac{12q^{78} P(1)^3 P(3)^2 P(2)^2}{P(5) P(4) P(6)^2} + \frac{10q^{91} P(1)^6 P(4)}{P(5) P(3) P(6)^2} + \frac{3q^{91} P(1)^4 P(2)^3}{P(5) P(4) P(6)^2} \\
& + \frac{3q^{104} P(1)^8}{P(5) P(2)^2 P(6)^2},
\end{aligned}$$

$$\begin{aligned}
A_{13,3}(q^{13}) &= \frac{3P(4)^7P(3)}{P(1)^2P(5)P(6)^2} + \frac{8q^{13}P(4)^4P(3)^3P(2)}{P(1)^2P(5)P(6)^2} + \frac{5q^{26}P(4)^3P(3)^4}{P(5)P(6)^2P(2)} + \frac{2q^{26}P(3)^8}{P(1)P(5)P(6)^2P(2)} + \frac{10q^{39}P(4)P(3)^3P(2)^3}{P(1)P(5)P(6)^2} \\
&+ \frac{10q^{39}P(3)^6}{P(5)P(6)^2} + \frac{11q^{52}P(4)P(3)P(2)^4}{P(5)P(6)^2} + \frac{5q^{52}P(1)P(3)^4P(2)}{P(5)P(6)^2} + \frac{q^{52}P(3)^2P(2)^7}{P(1)^2P(4)P(5)P(6)^2} + \frac{9q^{65}P(1)P(4)P(2)^5}{P(5)P(6)^2P(3)} \\
&+ \frac{5q^{65}P(1)^2P(3)^2P(2)^2}{P(5)P(6)^2} + \frac{12q^{65}P(2)^8}{P(1)P(4)P(5)P(6)^2} + \frac{10q^{78}P(1)^3P(2)^3}{P(5)P(6)^2} + \frac{11q^{78}P(1)^4P(3)^3}{P(4)P(5)P(6)^2} + \frac{5q^{91}P(1)^5P(3)P(2)}{P(4)P(5)P(6)^2} \\
&+ \frac{10q^{104}P(1)^6P(2)^2}{P(4)P(5)P(6)^2P(3)}, \\
A_{13,4}(q^{13}) &= \frac{5P(4)^8P(2)}{P(1)^2P(5)P(6)^2P(3)} + \frac{4q^{13}P(4)^4P(3)^4}{P(1)P(5)P(6)^2P(2)} + \frac{4q^{26}P(4)P(3)^6}{P(1)P(5)P(6)^2} + \frac{5q^{39}P(4)^2P(3)P(2)^4}{P(1)P(5)P(6)^2} + \frac{11q^{39}P(4)P(3)^4P(2)}{P(5)P(6)^2} \\
&+ \frac{7q^{52}P(2)^7}{P(1)P(6)^2P(3)} + \frac{11q^{52}P(1)P(4)P(3)^2P(2)^2}{P(5)P(6)^2} + \frac{7q^{52}P(1)^2P(3)^5}{P(5)P(6)^2P(2)} + \frac{6q^{52}P(2)^8}{P(1)^2P(5)P(6)^2} + \frac{4q^{52}P(3)^3P(2)^5}{P(1)P(4)P(5)P(6)^2} \\
&+ \frac{5q^{65}P(1)^4P(4)^3}{P(5)P(6)^2P(3)} + \frac{2q^{65}P(1)^2P(4)P(2)^3}{P(5)P(6)^2} + \frac{7q^{65}P(1)^3P(3)^3}{P(5)P(6)^2} + \frac{9q^{65}P(3)P(2)^6}{P(4)P(5)P(6)^2} + \frac{3q^{78}P(1)^4P(3)P(2)}{P(5)P(6)^2} \\
&+ \frac{7q^{78}P(1)P(2)^7}{P(4)P(5)P(6)^2P(3)} + \frac{7q^{91}P(1)^5P(2)^2}{P(5)P(6)^2P(3)} + \frac{6q^{104}P(1)^7}{P(4)P(5)P(6)^2}, \\
A_{13,5}(q^{13}) &= \frac{3P(4)^9}{q^{13}P(5)^2P(6)P(1)^2P(2)} + \frac{11P(4)^6P(3)^2}{P(5)^2P(6)P(1)^2} + \frac{8q^{13}P(4)^2P(3)^7}{P(5)^2P(6)P(1)P(2)^2} + \frac{11q^{26}P(4)^2P(3)^5}{P(5)^2P(6)P(2)} + \frac{q^{26}P(3)^6P(2)^2}{P(5)^2P(6)P(1)^2} \\
&+ \frac{3q^{26}P(3)^9}{P(5)^2P(4)P(6)P(1)P(2)} + \frac{q^{39}P(4)P(3)P(2)^6}{P(5)^2P(6)P(1)^2} + \frac{9q^{39}P(3)^4P(2)^3}{P(5)^2P(6)P(1)} + \frac{5q^{39}P(3)^7}{P(5)^2P(4)P(6)} + \frac{12q^{52}P(4)P(2)^7}{P(5)^2P(6)P(1)P(3)} \\
&+ \frac{12q^{52}P(3)^2P(2)^4}{P(5)^2P(6)} + \frac{10q^{65}P(1)P(2)^5}{P(5)^2P(6)} + \frac{9q^{65}P(1)^2P(3)^3P(2)^2}{P(5)^2P(4)P(6)} + \frac{4q^{78}P(1)^3P(3)P(2)^3}{P(5)^2P(4)P(6)} + \frac{12q^{91}P(1)^4P(2)^4}{P(5)^2P(4)P(6)P(3)} \\
&+ \frac{10q^{104}P(1)^8}{P(5)^2P(6)P(3)P(2)}, \\
A_{13,6}(q^{13}) &= \frac{5P(3)^7P(4)^3}{P(1)^2P(6)P(5)^2P(2)^2} + \frac{q^{13}P(3)^5P(4)^3}{P(1)P(6)P(5)^2P(2)} + \frac{8q^{13}P(3)^9}{P(1)^2P(6)P(5)^2P(2)} + \frac{5q^{26}P(3)^7}{P(1)P(6)P(5)^2} + \frac{9q^{39}P(1)P(3)^8}{P(4)P(6)P(5)^2P(2)^2} \\
&+ \frac{7q^{39}P(3)^6P(2)^4}{P(1)^2P(4)^2P(6)P(5)^2} + \frac{3q^{52}P(1)P(3)^3P(2)^2}{P(6)P(5)^2} + \frac{3q^{52}P(1)^2P(3)^6}{P(4)P(6)P(5)^2P(2)} + \frac{4q^{52}P(3)^4P(2)^5}{P(1)P(4)^2P(6)P(5)^2} \\
&+ \frac{12q^{65}P(1)^2P(3)P(2)^3}{P(6)P(5)^2} + \frac{12q^{65}P(1)^3P(3)^4}{P(4)P(6)P(5)^2} + \frac{2q^{78}P(1)^3P(2)^4}{P(3)P(6)P(5)^2} + \frac{2q^{78}P(1)P(2)^7}{P(4)^2P(6)P(5)^2} + \frac{5q^{91}P(1)^5P(2)^2}{P(4)P(6)P(5)^2} \\
&+ \frac{4q^{91}P(1)^6P(3)^3}{P(4)^2P(6)P(5)^2P(2)} + \frac{6q^{104}P(1)^7P(3)}{P(4)^2P(6)P(5)^2}, \\
A_{13,7}(q^{13}) &= \frac{8P(3)^4P(4)^6}{P(1)^2P(6)^2P(5)^2P(2)} + \frac{3q^{13}P(3)^6P(4)^3}{P(1)^2P(6)^2P(5)^2} + \frac{7q^{26}P(3)^4P(2)P(4)^3}{P(1)P(6)^2P(5)^2} + \frac{2q^{26}P(3)^8P(2)}{P(1)^2P(6)^2P(5)^2} + \frac{8q^{39}P(1)P(3)^5P(4)^2}{P(6)^2P(5)^2P(2)} \\
&+ \frac{12q^{39}P(3)^3P(2)^5P(4)}{P(1)^2P(6)^2P(5)^2} + \frac{7q^{39}P(3)^6P(2)^2}{P(1)P(6)^2P(5)^2} + \frac{9q^{52}P(3)P(2)^6P(4)}{P(1)P(6)^2P(5)^2} + \frac{9q^{52}P(3)^4P(2)^3}{P(6)^2P(5)^2} + \frac{9q^{65}P(1)P(3)^2P(2)^4}{P(6)^2P(5)^2} \\
&+ \frac{5q^{65}P(1)^2P(3)^5P(2)}{P(6)^2P(5)^2P(4)} + \frac{6q^{78}P(1)^3P(3)^3P(2)^2}{P(6)^2P(5)^2P(4)} + \frac{12q^{91}P(1)^4P(3)P(2)^3}{P(6)^2P(5)^2P(4)}, \\
A_{13,8}(q^{13}) &= \frac{4P(3)^6P(4)^2}{P(1)^2P(5)P(2)P(6)} + \frac{4q^{13}P(3)^4P(4)^2}{P(1)P(5)P(6)} + \frac{9q^{13}P(3)^8}{P(1)^2P(5)P(6)P(4)} + \frac{10q^{26}P(2)P(3)^2P(4)^2}{P(5)P(6)} + \frac{3q^{26}P(2)^4P(3)^3}{P(1)^2P(5)P(6)} \\
&+ \frac{6q^{26}P(2)P(3)^6}{P(1)P(5)P(6)P(4)} + \frac{4q^{39}P(2)^4}{P(6)} + \frac{4q^{39}P(1)^2P(3)^3P(4)}{P(5)P(2)P(6)} + \frac{12q^{39}P(2)^5P(3)}{P(1)P(5)P(6)} + \frac{4q^{39}P(2)^2P(3)^4}{P(5)P(6)P(4)} \\
&+ \frac{8q^{52}P(1)^3P(3)P(4)}{P(5)P(6)} + \frac{7q^{52}P(1)P(2)^3P(3)^2}{P(5)P(6)P(4)} + \frac{11q^{65}P(1)^2P(2)^4}{P(5)P(6)P(4)}, \\
A_{13,9}(q^{13}) &= \frac{10P(4)^2P(3)^8}{P(1)^2P(6)P(2)^2P(5)^2} + \frac{10q^{13}P(4)^2P(3)^6}{P(1)P(6)P(2)P(5)^2} + \frac{3q^{13}P(3)^{10}}{P(1)^2P(6)P(4)P(2)P(5)^2} + \frac{10q^{26}P(4)^2P(3)^4}{P(6)P(5)^2} \\
&+ \frac{3q^{26}P(3)^8}{P(1)P(6)P(4)P(5)^2} + \frac{11q^{39}P(2)^4P(3)^3}{P(1)P(6)P(5)^2} + \frac{11q^{39}P(2)P(3)^6}{P(6)P(4)P(5)^2} + \frac{11q^{52}P(2)^5P(3)}{P(6)P(5)^2} + \frac{9q^{52}P(1)P(2)^2P(3)^4}{P(6)P(4)P(5)^2} \\
&+ \frac{7q^{52}P(2)^8P(3)^2}{P(1)^2P(6)P(4)^2P(5)^2} + \frac{6q^{65}P(1)P(2)^6}{P(6)P(5)^2P(3)} + \frac{10q^{65}P(1)^2P(2)^3P(3)^2}{P(6)P(4)P(5)^2} + \frac{10q^{65}P(1)^3P(3)^5}{P(6)P(4)^2P(5)^2} \\
&+ \frac{6q^{65}P(2)^9}{P(1)P(6)P(4)^2P(5)^2} + \frac{5q^{78}P(1)^3P(2)^4}{P(6)P(4)P(5)^2} + \frac{3q^{78}P(1)^4P(2)P(3)^3}{P(6)P(4)^2P(5)^2} + \frac{2q^{91}P(1)^5P(2)^2P(3)}{P(6)P(4)^2P(5)^2}, \\
A_{13,10}(q^{13}) &= \frac{8P(4)^3P(3)^6}{P(1)^2P(2)P(6)P(5)^2} + \frac{q^{13}P(4)^3P(3)^4}{P(1)P(6)P(5)^2} + \frac{5q^{13}P(3)^8}{P(1)^2P(6)P(5)^2} + \frac{2q^{26}P(2)P(3)^6}{P(1)P(6)P(5)^2} + \frac{7q^{39}P(2)^2P(3)^4}{P(6)P(5)^2} \\
&+ \frac{7q^{39}P(1)P(3)^7}{P(4)P(2)P(6)P(5)^2} + \frac{10q^{39}P(2)^5P(3)^5}{P(4)^2P(1)^2P(6)P(5)^2} + \frac{3q^{52}P(1)P(2)^3P(3)^2}{P(6)P(5)^2} + \frac{7q^{52}P(1)^2P(3)^5}{P(4)P(6)P(5)^2} \\
&+ \frac{12q^{52}P(2)^6P(3)^3}{P(4)^2P(1)P(6)P(5)^2} + \frac{5q^{65}P(1)^2P(2)^4}{P(6)P(5)^2} + \frac{7q^{65}P(1)^3P(2)P(3)^3}{P(4)P(6)P(5)^2} + \frac{4q^{65}P(2)^7P(3)}{P(4)^2P(6)P(5)^2} + \frac{10q^{78}P(1)^4P(2)^2P(3)}{P(4)P(6)P(5)^2} \\
&+ \frac{6q^{91}P(1)^6P(3)^2}{P(4)^2P(6)P(5)^2}, \\
A_{13,11}(q^{13}) &= \frac{5P(3)P(4)^5}{q^{13}P(1)^2P(6)} + \frac{5q^{13}P(3)^5P(4)}{P(1)P(5)P(6)} + \frac{12q^{26}P(2)P(3)^3P(4)}{P(5)P(6)} + \frac{9q^{26}P(2)^4P(3)^4}{P(1)^2P(5)P(4)P(6)} + \frac{7q^{26}P(2)P(3)^7}{P(1)P(5)P(4)^2P(6)} \\
&+ \frac{6q^{39}P(2)^8}{P(3)P(1)^2P(5)P(6)} + \frac{5q^{39}P(2)^5P(3)^2}{P(1)P(5)P(4)P(6)} + \frac{3q^{39}P(2)^2P(3)^5}{P(5)P(4)^2P(6)} + \frac{6q^{52}P(3)^2P(1)^3}{P(5)P(6)} + \frac{5q^{52}P(2)^3P(3)^3P(1)}{P(5)P(4)^2P(6)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{11q^{65}P(2)^4P(3)P(1)^2}{P(5)P(4)^2P(6)} + \frac{6q^{78}P(2)^5P(1)^3}{P(3)P(5)P(4)^2P(6)}, \\
A_{13,12}(q^{13}) &= \frac{2P(4)^2P(3)^5}{P(1)^2P(5)P(6)} + \frac{4q^{13}P(4)P(3)^6}{P(2)^2P(5)P(6)} + \frac{11q^{13}P(3)^7P(2)}{P(4)P(1)^2P(5)P(6)} + \frac{5q^{26}P(2)^2P(4)^2P(3)}{P(5)P(6)} + \frac{4q^{26}P(4)P(3)^4P(1)}{P(2)P(5)P(6)} \\
& + \frac{4q^{26}P(3)^2P(2)^5}{P(1)^2P(5)P(6)} + \frac{5q^{26}P(3)^5P(2)^2}{P(4)P(1)P(5)P(6)} + \frac{q^{39}P(2)^5}{P(3)P(6)} + \frac{10q^{39}P(4)P(3)^2P(1)^2}{P(5)P(6)} + \frac{10q^{39}P(2)^6}{P(1)P(5)P(6)} \\
& + \frac{12q^{39}P(3)^3P(2)^3}{P(4)P(5)P(6)} + \frac{4q^{52}P(4)P(1)^3P(2)}{P(5)P(6)} + \frac{9q^{52}P(3)P(1)P(2)^4}{P(4)P(5)P(6)} + \frac{9q^{65}P(3)P(1)^5}{P(2)P(5)P(6)} + \frac{q^{65}P(1)^2P(2)^5}{P(4)P(3)P(5)P(6)}, \\
B_{13,0}(q^{13}) &= \frac{6q^{13}P(4)^5P(3)^2P(2)}{P(1)^2P(6)^2P(5)} + \frac{7q^{26}P(4)P(3)^7}{P(1)P(6)^2P(2)P(5)} + \frac{6q^{39}P(4)P(3)^5}{P(6)^2P(5)} + \frac{10q^{39}P(3)^6P(2)^3}{P(4)P(1)^2P(6)^2P(5)} + \frac{3q^{39}P(3)^9}{P(4)^2P(1)P(6)^2P(5)} \\
& + \frac{2q^{52}P(3)P(2)^7}{P(1)^2P(6)^2P(5)} + \frac{12q^{52}P(3)^4P(2)^4}{P(4)P(1)P(6)^2P(5)} + \frac{6q^{52}P(3)^7P(2)}{P(4)^2P(6)^2P(5)} + \frac{10q^{65}P(1)P(3)^5P(2)^2}{P(4)^2P(6)^2P(5)} + \frac{12q^{78}P(1)^2P(3)^3P(2)^3}{P(4)^2P(6)^2P(5)} \\
& + \frac{7q^{91}P(1)^3P(3)P(2)^4}{P(4)^2P(6)^2P(5)} + \frac{3q^{104}P(1)^6P(2)^2}{P(3)^2P(6)^2P(5)} + \frac{3q^{104}P(1)^4P(2)^5}{P(4)^2P(3)P(6)^2P(5)} + \frac{10q^{117}P(1)^8}{P(4)P(3)P(6)^2P(5)}, \\
B_{13,1}(q^{13}) &= \frac{5q^{13}P(4)^3P(3)^4}{P(6)P(1)P(5)P(2)} + \frac{9q^{26}P(4)^3P(3)^2}{P(6)P(5)} + \frac{12q^{26}P(4)P(3)^3P(2)^3}{P(6)P(1)^2P(5)} + \frac{9q^{26}P(3)^6}{P(6)P(1)P(5)} + \frac{10q^{39}P(4)P(3)P(2)^4}{P(6)P(1)P(5)} \\
& + \frac{7q^{39}P(3)^4P(2)}{P(6)P(5)} + \frac{10q^{52}P(1)P(3)^2P(2)^2}{P(6)P(5)} + \frac{4q^{65}P(1)^2P(2)^3}{P(6)P(5)} + \frac{7q^{78}P(1)^4P(3)P(2)}{P(6)P(4)P(5)}, \\
B_{13,2}(q^{13}) &= \frac{P(3)^3P(4)^6}{P(6)^2P(2)P(1)^2P(5)} + \frac{q^{13}P(3)^5P(4)^3}{P(6)^2P(1)^2P(5)} + \frac{7q^{26}P(3)^6P(4)^2}{P(6)^2P(2)^2P(5)} + \frac{2q^{26}P(2)P(3)^7}{P(6)^2P(1)^2P(5)} + \frac{9q^{26}P(3)^{10}}{P(6)^2P(2)^2P(1)P(4)P(5)} \\
& + \frac{11q^{39}P(2)^5P(3)^2P(4)}{P(6)^2P(1)^2P(5)} + \frac{8q^{39}P(2)^2P(3)^5}{P(6)^2P(1)P(5)} + \frac{6q^{52}P(2)^6P(4)}{P(6)^2P(1)P(5)} + \frac{6q^{52}P(2)^3P(3)^3}{P(6)^2P(5)} + \frac{10q^{52}P(1)P(3)^6}{P(6)^2P(4)P(5)} \\
& + \frac{8q^{65}P(2)^4P(1)P(3)}{P(6)^2P(5)} + \frac{q^{65}P(2)P(1)^2P(3)^4}{P(6)^2P(4)P(5)} + \frac{q^{78}P(2)^2P(1)^3P(3)^2}{P(6)^2P(4)P(5)} + \frac{9q^{91}P(1)^6P(4)}{P(6)^2P(3)P(5)} + \frac{2q^{91}P(2)^3P(1)^4}{P(6)^2P(4)P(5)} \\
& + \frac{4q^{104}P(1)^8}{P(6)^2P(2)^2P(5)}, \\
B_{13,3}(q^{13}) &= \frac{6P(3)P(4)^7}{P(6)^2P(1)^2P(5)} + \frac{3q^{13}P(2)P(3)^3P(4)^4}{P(6)^2P(1)^2P(5)} + \frac{4q^{26}P(3)^8}{P(2)P(6)^2P(1)P(5)} + \frac{10q^{26}P(3)^4P(4)^3}{P(2)P(6)^2P(5)} + \frac{7q^{39}P(2)^3P(3)^3P(4)}{P(6)^2P(1)P(5)} \\
& + \frac{7q^{39}P(3)^6}{P(6)^2P(5)} + \frac{9q^{52}P(2)^4P(3)P(4)}{P(6)^2P(5)} + \frac{10q^{52}P(2)P(3)^4P(1)}{P(6)^2P(5)} + \frac{2q^{52}P(2)^7P(3)^2}{P(6)^2P(1)^2P(5)P(4)} + \frac{5q^{65}P(2)^5P(1)P(4)}{P(3)P(6)^2P(5)} \\
& + \frac{10q^{65}P(2)^2P(3)^2P(1)^2}{P(6)^2P(5)} + \frac{11q^{65}P(2)^8}{P(6)^2P(1)P(5)P(4)} + \frac{7q^{78}P(2)^3P(1)^3}{P(6)^2P(5)} + \frac{9q^{78}P(3)^3P(1)^4}{P(6)^2P(5)P(4)} + \frac{10q^{91}P(2)P(3)P(1)^5}{P(6)^2P(5)P(4)} \\
& + \frac{7q^{104}P(2)^2P(1)^6}{P(3)P(6)^2P(5)P(4)}, \\
B_{13,4}(q^{13}) &= \frac{2P(2)P(4)^8}{P(3)P(6)^2P(1)^2P(5)} + \frac{q^{13}P(3)^4P(4)^4}{P(2)P(6)^2P(1)P(5)} + \frac{4q^{26}P(3)^6P(4)}{P(6)^2P(1)P(5)} + \frac{12q^{39}P(2)P(3)^4P(4)}{P(6)^2P(5)} + \frac{6q^{39}P(2)^4P(3)^5}{P(6)^2P(1)^2P(5)P(4)} \\
& + \frac{10q^{52}P(2)^2P(3)^2P(1)P(4)}{P(6)^2P(5)} + \frac{11q^{52}P(3)^5P(1)^2}{P(2)P(6)^2P(5)} + \frac{q^{52}P(2)^5P(3)^3}{P(6)^2P(1)P(5)P(4)} + \frac{12q^{65}P(2)^3P(1)^2P(4)}{P(6)^2P(5)} \\
& + \frac{10q^{65}P(2)^6P(3)}{P(6)^2P(5)P(4)} + \frac{7q^{78}P(2)^4P(1)^3P(4)}{P(3)^2P(6)^2P(5)} + \frac{2q^{78}P(2)P(3)P(1)^4}{P(6)^2P(5)} + \frac{9q^{78}P(2)^7P(1)}{P(3)P(6)^2P(5)P(4)} + \frac{4q^{91}P(2)^2P(1)^5}{P(3)P(6)^2P(5)} \\
& + \frac{2q^{104}P(1)^7}{P(6)^2P(5)P(4)} + \frac{11q^{117}P(2)P(1)^8}{P(3)^2P(6)^2P(5)P(4)}, \\
B_{13,5}(q^{13}) &= \frac{10P(4)^6P(3)^3}{q^{13}P(6)P(1)^2P(5)P(2)^2} + \frac{3P(4)^3P(3)^5}{P(6)P(1)^2P(5)P(2)} + \frac{2q^{13}P(4)^3P(3)^3}{P(6)P(1)P(5)} + \frac{10q^{26}P(1)P(4)^2P(3)^4}{P(6)P(5)P(2)^2} \\
& + \frac{12q^{26}P(4)P(3)^2P(2)^4}{P(6)P(1)^2P(5)} + \frac{12q^{26}P(3)^5P(2)}{P(6)P(1)P(5)} + \frac{5q^{39}P(4)P(2)^5}{P(6)P(1)P(5)} + \frac{11q^{39}P(3)^3P(2)^2}{P(6)P(5)} + \frac{q^{52}P(1)P(3)P(2)^3}{P(6)P(5)} \\
& + \frac{3q^{52}P(1)^2P(3)^4}{P(6)P(4)P(5)} + \frac{3q^{65}P(1)^3P(3)^2P(2)}{P(6)P(4)P(5)} + \frac{8q^{78}P(1)^4P(2)^2}{P(6)P(4)P(5)}, \\
B_{13,6}(q^{13}) &= \frac{P(3)^3P(4)^4}{P(5)P(6)P(1)^2} + \frac{7q^{13}P(3)^8}{P(5)P(6)P(2)^2P(1)} + \frac{8q^{26}P(3)^6}{P(5)P(6)P(2)} + \frac{5q^{26}P(3)^7P(2)^2}{P(5)P(4)^2P(6)P(1)^2} + \frac{6q^{39}P(3)^2P(2)^6}{P(5)P(4)P(6)P(1)^2} \\
& + \frac{12q^{39}P(3)^5P(2)^3}{P(5)P(4)^2P(6)P(1)} + \frac{5q^{39}P(3)^4P(1)}{P(5)P(6)} + \frac{3q^{52}P(2)^6}{P(3)P(4)P(6)} + \frac{9q^{52}P(3)^5P(1)^3}{P(5)P(4)P(6)P(2)^2} + \frac{4q^{52}P(2)^7}{P(5)P(4)P(6)P(1)} \\
& + \frac{11q^{52}P(3)^3P(2)^4}{P(5)P(4)^2P(6)} + \frac{2q^{65}P(2)^2P(1)^3}{P(5)P(6)} + \frac{2q^{65}P(3)^3P(1)^4}{P(5)P(4)P(6)P(2)} + \frac{12q^{65}P(3)P(2)^5P(1)}{P(5)P(4)^2P(6)} + \frac{12q^{78}P(2)^3P(1)^4}{P(5)P(3)^2P(6)} \\
& + \frac{12q^{78}P(3)P(1)^5}{P(5)P(4)P(6)} + \frac{2q^{78}P(2)^6P(1)^2}{P(5)P(3)P(4)^2P(6)} + \frac{10q^{91}P(2)P(1)^6}{P(5)P(3)P(4)P(6)} + \frac{4q^{104}P(1)^8}{P(5)P(4)^2P(6)P(2)}, \\
B_{13,7}(q^{13}) &= \frac{4P(2)P(4)^7}{P(6)^2P(1)^2P(5)} + \frac{8q^{13}P(3)^5P(4)^3}{P(6)^2P(1)P(5)P(2)} + \frac{3q^{26}P(3)^4P(2)^3P(4)}{P(6)^2P(1)^2P(5)} + \frac{11q^{26}P(3)^7}{P(6)^2P(1)P(5)} + \frac{4q^{39}P(3)^2P(2)^4P(4)}{P(6)^2P(1)P(5)} \\
& + \frac{6q^{39}P(3)^5P(2)}{P(6)^2P(5)} + \frac{10q^{52}P(5)P(2)^6}{P(6)^2P(5)} + \frac{9q^{52}P(1)P(3)^3P(2)^2}{P(6)^2P(5)} + \frac{3q^{52}P(3)P(2)^8}{P(6)^2P(1)^2P(5)P(4)} + \frac{4q^{65}P(1)^4P(4)^2}{P(6)^2P(5)} \\
& + \frac{6q^{65}P(1)^2P(3)P(2)^3}{P(6)^2P(5)} + \frac{10q^{65}P(2)^9}{P(6)^2P(1)P(3)P(5)P(4)} + \frac{2q^{78}P(1)^4P(3)^2P(2)}{P(6)^2P(5)P(4)} + \frac{10q^{91}P(1)^5P(2)^2}{P(6)^2P(5)P(4)}, \\
B_{13,8}(q^{13}) &= \frac{10P(3)^6P(4)^2}{P(1)^2P(6)P(5)P(2)} + \frac{3q^{13}P(3)^4P(4)^2}{P(1)P(6)P(5)} + \frac{3q^{13}P(3)^8}{P(1)^2P(4)P(6)P(5)} + \frac{8q^{26}P(3)^2P(4)^2P(2)}{P(6)P(5)} + \frac{9q^{26}P(3)^3P(2)^4}{P(1)^2P(6)P(5)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q^{26}P(3)^6P(2)}{P(1)P(4)P(6)P(5)} + \frac{12q^{39}P(2)^4}{P(6)} + \frac{10q^{39}P(1)^2P(3)^3P(4)}{P(6)P(5)P(2)} + \frac{10q^{39}P(3)P(2)^5}{P(1)P(6)P(5)} + \frac{8q^{39}P(3)^4P(2)^2}{P(4)P(6)P(5)} \\
& + \frac{q^{52}P(1)^3P(3)P(4)}{P(6)P(5)} + \frac{6q^{52}P(1)P(3)^2P(2)^3}{P(4)P(6)P(5)} + \frac{2q^{65}P(1)^2P(2)^4}{P(4)P(6)P(5)}, \\
B_{13,9}(q^{13}) &= \frac{8P(3)^4P(4)^3}{P(5)P(6)P(1)^2} + \frac{10q^{13}P(3)^5P(4)^2}{P(5)P(6)P(2)^2} + \frac{5q^{13}P(3)^9}{P(5)P(6)P(1)P(4)P(2)^2} + \frac{9q^{26}P(3)^7}{P(5)P(6)P(4)P(2)} + \frac{9q^{39}P(3)^2P(2)^3}{P(5)P(6)} \\
& + \frac{12q^{39}P(1)P(3)^5}{P(5)P(6)P(4)} + \frac{7q^{39}P(3)^3P(2)^6}{P(5)P(6)P(1)^2P(4)^2} + \frac{4q^{52}P(1)P(2)^4}{P(5)P(6)} + \frac{9q^{52}P(1)^2P(3)^3P(2)}{P(5)P(6)P(4)} + \frac{10q^{52}P(3)P(2)^7}{P(5)P(6)P(1)P(4)^2} \\
& + \frac{9q^{65}P(1)^2P(2)^5}{P(5)P(6)P(3)^2} + \frac{2q^{65}P(1)^3P(3)P(2)^2}{P(5)P(6)P(4)} + \frac{8q^{65}P(1)^4P(3)^4}{P(5)P(6)P(4)^2P(2)} + \frac{9q^{65}P(2)^8}{P(5)P(6)P(3)P(4)^2} + \frac{4q^{78}P(1)^4P(2)^3}{P(5)P(6)P(3)P(4)} \\
& + \frac{9q^{78}P(1)^5P(3)^2}{P(5)P(6)P(4)^2} + \frac{8q^{91}P(1)^6P(2)}{P(5)P(6)P(4)^2}, \\
B_{13,10}(q^{13}) &= 0, \\
B_{13,11}(q^{13}) &= \frac{4P(3)P(4)^5}{q^{13}P(6)P(1)^2} + \frac{4q^{13}P(3)^5P(4)}{P(6)P(1)P(5)} + \frac{7q^{26}P(3)^3P(2)P(4)}{P(6)P(5)} + \frac{2q^{26}P(3)^4P(2)^4}{P(6)P(1)^2P(4)P(5)} + \frac{3q^{26}P(3)^7P(2)}{P(6)P(1)P(4)^2P(5)} \\
& + \frac{10q^{39}P(2)^8}{P(6)P(1)^2P(3)P(5)} + \frac{4q^{39}P(3)^2P(2)^5}{P(6)P(1)P(4)P(5)} + \frac{5q^{39}P(3)^5P(2)^2}{P(6)P(4)^2P(5)} + \frac{10q^{52}P(1)^3P(3)^2}{P(6)P(5)} + \frac{4q^{52}P(1)P(3)^3P(2)^3}{P(6)P(4)^2P(5)} \\
& + \frac{q^{65}P(1)^2P(3)P(2)^4}{P(6)P(4)^2P(5)} + \frac{10q^{78}P(1)^3P(2)^5}{P(6)P(3)P(4)^2P(5)}, \\
B_{13,12}(q^{13}) &= \frac{9P(4)^2P(3)^5}{P(1)^2P(6)P(5)} + \frac{5q^{13}P(4)P(3)^6}{P(6)P(2)^2P(5)} + \frac{4q^{13}P(2)P(3)^7}{P(1)^2P(4)P(6)P(5)} + \frac{8q^{26}P(1)P(4)P(3)^4}{P(6)P(2)P(5)} + \frac{5q^{26}P(2)^5P(3)^2}{P(1)^2P(6)P(5)} \\
& + \frac{3q^{26}P(2)^2P(3)^5}{P(1)P(4)P(6)P(5)} + \frac{11q^{39}P(2)^5}{P(6)P(3)} + \frac{9q^{39}P(1)^2P(4)P(3)^2}{P(6)P(5)} + \frac{6q^{39}P(2)^6}{P(1)P(6)P(5)} + \frac{2q^{39}P(2)^3P(3)^3}{P(4)P(6)P(5)} \\
& + \frac{5q^{52}P(1)^3P(4)P(2)}{P(6)P(5)} + \frac{5q^{52}P(1)P(2)^4P(3)}{P(4)P(6)P(5)} + \frac{3q^{65}P(1)^4P(4)P(2)^2}{P(6)P(5)P(3)^2} + \frac{8q^{65}P(1)^5P(3)}{P(6)P(2)P(5)} + \frac{q^{65}P(1)^2P(2)^5}{P(4)P(6)P(5)P(3)} \\
& + \frac{10q^{78}P(1)^6}{P(6)P(5)P(3)}.
\end{aligned}$$

Theorem 1.2 implies Theorem 1.1 as the coefficients of the powers of  $q$  for the appropriate residue classes are zero in the above congruences. It is worth noting that none of the representations of  $U(q)$  and  $V(q)$  that we will use suggest that these functions are even congruent to modular forms, whereas  $F(q)$  is essentially a modular form. There are few partition functions that satisfy such simple congruences modulo 13 that are not modular forms or closely related to modular forms. However, we can recognize the series appearing on the right hand side of Theorem 1.2 as mock modular forms. For example, in the notation of [14] we can write

$$\frac{q^2}{E(9)P(1)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{9n^2+15n}{2}}}{(1-q^{9n+3})} = iq^{\frac{1}{8}} \mu(3\tau, 3\tau; 9\tau),$$

where  $q = e^{2\pi i\tau}$ . With this the functions  $U(q)$  and  $V(q)$  should be quasimock mock modular forms, which one should consult [6] to understand. To establish these functions are indeed quasimock theta functions one should use the identities in the next section to recognize  $U(q)$  and  $V(q)$  as a second derivative of a certain Lambert series.

In Section 2 we develop the general identities and congruences necessary to prove Theorem 1.2. While we will only use these congruences modulo 3, 5, 7, and 13, they hold for all odd  $\ell > 1$  (in particular  $\ell$  need not be prime). In this way, the identities of the article could also be used to determine  $U(q)$ ,  $V(q)$ , and related functions to other moduli. In Section 3 we prove Theorem 1.2 and in Section 4 we give a few concluding remarks.

## 2. PRELIMINARY IDENTITIES

To begin we find another form of the generating functions for  $U(q)$  and  $V(q)$ . We say a pair of sequences  $(\alpha, \beta)$  is a Bailey pair relative to  $(a, q)$  if

$$\beta_n = \sum_{k=0}^{\infty} \frac{\alpha_k}{(q; q)_{n-k} (aq; q)_{n+k}}.$$

A limiting case of Bailey's Lemma gives that

$$\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left( \frac{aq}{\rho_1 \rho_2} \right)^n \beta_n = \frac{(aq/\rho_1, aq/\rho_2; q)_{\infty}}{\left( aq, \frac{aq}{\rho_1 \rho_2}; q \right)_{\infty}} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n \left( \frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n}{(aq/\rho_1, aq/\rho_2; q)_n}.$$

One may consult [2] for a history of Bailey pairs and Bailey's Lemma. When  $(\alpha, \beta)$  is relative to  $(1, q)$  and we set  $\rho_1 = z$ ,  $\rho_2 = z^{-1}$  this reduces to

$$\sum_{n=0}^{\infty} (z, z^{-1}; q)_n q^n \beta_n = \frac{(z, z^{-1}; q)_{\infty}}{(q; q)_{\infty}^2} \sum_{n=0}^{\infty} \frac{q^n \alpha_n}{(1 - zq^n)(1 - z^{-1}q^n)}.$$

We recall one version of the finite Jacobi triple product identity [1, page 49] is

$$\frac{(xq, x^{-1}; q)_n}{(q; q)_{2n}} = \sum_{j=-n}^n \frac{(-1)^j x^j q^{j(j+1)/2}}{(q; q)_{n-j} (q; q)_{n+j}} = \frac{1}{(q; q)_{2n}} + \sum_{j=1}^n \frac{(-1)^j q^{j(j-1)/2} (x^{-n} + x^n q^n)}{(q; q)_{n-j} (q; q)_{n+j}}.$$

In the language of Bailey pairs, this says that  $(\alpha'(x), \beta'(x))$  is a Bailey pair with respect to  $(1, q)$ , where

$$\beta'_n(x) = \frac{(xq, x^{-1}; q)_n}{(q; q)_{2n}},$$

$$\alpha'_n(x) = \begin{cases} 1 & n = 0 \\ (-1)^n q^{n(n-1)/2} (x^{-n} + x^n q^n) & n \geq 1 \end{cases}.$$

A direct, but somewhat lengthy, calculation shows that

$$\frac{d^2}{dx^2} \beta'_n(x)|_{x=1} = -2 \frac{(q; q)_{n-1}^2}{(q; q)_{2n}},$$

$$\frac{d^2}{dx^2} \beta'_n(x)|_{x=q^{-1}} = -2q^{n+2} \frac{(q; q)_{n-1}^2}{(q; q)_{2n}}.$$

This gives the Bailey pairs, with respect to  $(1, q)$ ,

$$\beta_n^u = \frac{(q; q)_{n-1}^2}{(q; q)_{2n}}, \quad \alpha_n^u = (-1)^{n+1} q^{n(n-1)/2} \left( \frac{n(n+1)}{2} + \frac{n(n-1)}{2} q^n \right),$$

$$\beta_n^v = \frac{q^n (q; q)_{n-1}^2}{(q; q)_{2n}}, \quad \alpha_n^v = (-1)^{n+1} q^{n(n-1)/2} \left( \frac{n(n+1)}{2} q^n + \frac{n(n-1)}{2} \right).$$

We note  $\beta_0^u = \beta_0^v = \alpha_0^u = \alpha_0^v = 0$ .

By Bailey's Lemma, we then have that

$$\begin{aligned} U(q) &= \frac{1}{(q; q)_{\infty}^3} \sum_{n=1}^{\infty} \frac{(q; q)_{n-1}^4 q^n}{(q; q)_{2n}} \\ &= \frac{1}{(q, z, z^{-1}; q)_{\infty}} \sum_{n=1}^{\infty} \frac{(z, z^{-1}; q)_n q^n (q; q)_{n-1}^2}{(q; q)_{2n}} \Big|_{z=1} \\ &= \frac{1}{(q, z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n q^n \beta_n^u \Big|_{z=1} \\ &= \frac{1}{(q; q)_{\infty}^3} \sum_{n=0}^{\infty} \frac{q^n \alpha_n^u}{(1 - zq^n)(1 - z^{-1}q^n)} \Big|_{z=1} \\ &= \frac{-1}{2(q; q)_{\infty}^3} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (n(n+1) + n(n-1)q^n)}{(1 - zq^n)(1 - z^{-1}q^n)} \Big|_{z=1} \\ &= \frac{-1}{2(q; q)_{\infty}^3} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (n(n+1) + n(n-1)q^n)}{(1 - q^n)^2} \\ &= \frac{-1}{2(q; q)_{\infty}^3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} n(n+1)}{(1 - q^n)^2}. \end{aligned}$$

Here and elsewhere, a prime superscript in a summation indicates to omit the terms that correspond to a division by zero. Similarly we have that

$$V(q) = \frac{-1}{2(q; q)_\infty^3} \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} n(n-1)}{(1-q^n)^2}$$

We define the two series

$$S_\ell(b) = \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2 + bn} n(n+1)}{(1-q^{\ell n})},$$

$$T(z, w, q) = \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} w^n}{1 - zq^n}.$$

The series  $T(z, w, q)$  is a generalized Lambert series that appears quite often in number theory. It has been used by Lewis [10], Lewis and Santa-Gadea [11], and Ekin [8] in studying the crank of partition. Also up to a product, it is the Apell-Lerch sum studied in detail by Zweegers [14] and Hickerson and Mortenson [9].

To work with the series  $T(z, w, q)$ , we use the  $r = 1, s = 2$  case of Theorem 2.1 of [7], which is that

$$\frac{[a; q]_\infty (q; q)_\infty^2}{[b_1, b_2; q]_\infty} = \frac{[a/b_1; q]_\infty}{[b_2/b_1; q]_\infty} T(b_1, a/b_2, q) + \frac{[a/b_2; q]_\infty}{[b_1/b_2; q]_\infty} T(b_2, a/b_1, q). \quad (2.1)$$

Additionally, by letting  $n \mapsto -n + 1$ , we have that

$$T(z, w, q) = z^{-1} w^{-1} q T(z^{-1} q, w^{-1} q, q). \quad (2.2)$$

For integers  $a, b$ , and  $c$  we write

$$T(a, b, c) := T(q^a, q^b, q^c).$$

**Lemma 2.1.** *For  $\ell > 1$  odd and  $b$  any integer,*

$$S_\ell(b) \equiv \frac{2b(-1)^b q^{\ell m - \frac{b(b+1)}{2}} (q; q)_\infty^3}{(q^{\ell^2}; q^{\ell^2})_\infty [q^{\ell m}; q^{\ell^2}]_\infty} T\left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2\right)$$

$$+ (-1)^{\frac{\ell+1}{2} + b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \frac{(q^{\ell^2}; q^{\ell^2})_\infty^2}{[q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}; q^{\ell^2}]_\infty}$$

$$\times \sum_{\substack{k=0, \\ k \not\equiv b + \frac{1}{2} \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k-b-\frac{1}{2})(k-b+\frac{1}{2}) \frac{[q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}, q^{\ell k - \ell m}; q^{\ell^2}]_\infty}{[q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2}]_\infty} \pmod{\ell},$$

where  $m$  is any integer such that  $1 \leq m \leq \ell - 1$  and  $m \not\equiv b + \frac{1}{2} \pmod{\ell}$ .

*Proof.* In the series form of  $S_\ell(b)$ , we replace  $n$  by  $\ell n + k + c$ , where  $c = \frac{\ell-1}{2} - b$  and  $k = 0, 1, \dots, \ell - 1$ . Working modulo  $\ell$ , this gives that

$$S_\ell(b) = \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2} + bn} n(n+1)}{1 - q^{\ell n}}$$

$$\equiv \sum_{k=0}^{\ell-1} (-1)^{k+c} q^{\frac{c(c+1)}{2} + \frac{k(k+1)}{2} + ck + bc + bk} (k+c)(k+c+1) \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{\ell^2 n^2 + \ell n}{2} + \ell cn + \ell kn + \ell bn}}{1 - q^{\ell^2 n + \ell k + \ell c}}$$

$$\equiv \sum_{k=0}^{\ell-1} (-1)^{k+c} q^{\frac{c(c+1)}{2} + \frac{k(k+1)}{2} + ck + bc + bk} (k+c)(k+c+1) \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{\ell^2 n(n+1)}{2} + \ell(c+k+b-\frac{\ell-1}{2})n}}{1 - q^{\ell^2 n + \ell k + \ell c}}$$

$$\equiv \sum_{k=0}^{\ell-1} (-1)^{k+c} q^{\frac{c(c+1)}{2} + \frac{k(k+1)}{2} + ck + bc + bk} (k+c)(k+c+1) \sum'_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{\ell^2 n(n+1)}{2} + \ell kn}}{1 - q^{\ell^2 n + \ell k + \ell c}} \pmod{\ell}.$$



Here we note that the only division by zero would occur in the inner series when  $k \equiv -c \equiv b + \frac{1}{2} \pmod{\ell}$ . However, this entire term of the outer sum is zero modulo  $\ell$ , due to the factor of  $(k+c)$ . Additionally, by the restrictions on  $m$ , this does not correspond to the  $k = m$  term. Therefore we can omit the  $k \equiv -c \pmod{\ell}$  term and write the remaining inner series as  $T(\ell k + \ell c, \ell k, \ell^2)$ . With this in mind, we now have that

$$\begin{aligned}
S_\ell(b) &\equiv \sum_{\substack{k=0, \\ k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^{k+c} q^{\frac{c(c+1)}{2} + \frac{k(k+1)}{2} + ck + bc + bk} (k+c)(k+c+1) T(\ell k + \ell c, \ell k, \ell^2) \\
&\equiv (-1)^{\frac{\ell-1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2}} \sum_{\substack{k=0, \\ k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k+\ell)}{2}} (k+c)(k+c+1) T(\ell k + \ell c, \ell k, \ell^2) \\
&\equiv (-1)^{\frac{\ell-1}{2}+b+m} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \frac{m(m+\ell)}{2}} (m+c)(m+c+1) T(\ell m + \ell c, \ell m, \ell^2) \\
&\quad + (-1)^{\frac{\ell-1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2}} \sum_{\substack{k=0, \\ k \neq m, k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k+\ell)}{2}} (k+c)(k+c+1) T(\ell k + \ell c, \ell k, \ell^2) \pmod{\ell}.
\end{aligned}$$

For  $k \neq m$  and  $k \not\equiv -c \pmod{\ell}$ , applying (2.1) with  $q \mapsto q^{\ell^2}$ ,  $b_1 = q^{\ell m + \ell c}$ ,  $b_2 = q^{\ell k + \ell c}$ , and  $a = q^{\ell m + \ell k + \ell c}$  yields

$$\frac{(q^{\ell^2}; q^{\ell^2})_\infty^2 [q^{\ell m + \ell k + \ell c}; q^{\ell^2}]_\infty}{[q^{\ell m + \ell c}, q^{\ell k + \ell c}; q^{\ell^2}]_\infty} = \frac{[q^{\ell k}; q^{\ell^2}]_\infty}{[q^{\ell k - \ell m}; q^{\ell^2}]_\infty} T(\ell m + \ell c, \ell m, \ell^2) + \frac{[q^{\ell m}; q^{\ell^2}]_\infty}{[q^{\ell m - \ell k}; q^{\ell^2}]_\infty} T(\ell k + \ell c, \ell k, \ell^2),$$

which we rearrange to

$$q^{\ell k - \ell m} T(\ell k + \ell c, \ell k, \ell^2) = \frac{[q^{\ell k}; q^{\ell^2}]_\infty}{[q^{\ell m}; q^{\ell^2}]_\infty} T(\ell m + \ell c, \ell m, \ell^2) - \frac{(q^{\ell^2}; q^{\ell^2})_\infty^2 [q^{\ell m + \ell k + \ell c}, q^{\ell k - \ell m}; q^{\ell^2}]_\infty}{[q^{\ell m}, q^{\ell m + \ell c}, q^{\ell k + \ell c}; q^{\ell^2}]_\infty}.$$

The restrictions on  $m$  prevent a division by zero in the above identity. We note this identity also holds for  $k = m$ . Thus

$$\begin{aligned}
S_\ell(b) &\equiv T(\ell m + \ell c, \ell m, \ell^2) (-1)^{\frac{\ell-1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \sum_{\substack{k=0, \\ k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \frac{[q^{\ell k}; q^{\ell^2}]_\infty}{[q^{\ell m}; q^{\ell^2}]_\infty} \\
&\quad + (-1)^{\frac{\ell+1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \sum_{\substack{k=0, \\ k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \frac{(q^{\ell^2}; q^{\ell^2})_\infty^2 [q^{\ell m + \ell k + \ell c}, q^{\ell k - \ell m}; q^{\ell^2}]_\infty}{[q^{\ell m}, q^{\ell m + \ell c}, q^{\ell k + \ell c}; q^{\ell^2}]_\infty} \\
&\equiv T(\ell m + \ell c, \ell m, \ell^2) \frac{(-1)^{\frac{\ell-1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m}}{[q^{\ell m}; q^{\ell^2}]_\infty} \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) [q^{\ell k}; q^{\ell^2}]_\infty \\
&\quad + (-1)^{\frac{\ell+1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \frac{(q^{\ell^2}; q^{\ell^2})_\infty^2}{[q^{\ell m}, q^{\ell m + \ell c}; q^{\ell^2}]_\infty} \\
&\quad \times \sum_{\substack{k=0, \\ k \not\equiv -c \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \frac{[q^{\ell m + \ell k + \ell c}, q^{\ell k - \ell m}; q^{\ell^2}]_\infty}{[q^{\ell k + \ell c}; q^{\ell^2}]_\infty} \\
&\equiv T(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2) \frac{(-1)^{\frac{\ell-1}{2}+b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m}}{[q^{\ell m}; q^{\ell^2}]_\infty} \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) [q^{\ell k}; q^{\ell^2}]_\infty
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{\frac{\ell+1}{2}+b} q^{\frac{\ell^2-1}{8}-\frac{b(b+1)}{2}+\ell m} \frac{\left(q^{\ell^2}; q^{\ell^2}\right)_{\infty}^2}{\left[q^{\ell m}, q^{\frac{\ell(\ell-1)}{2}+\ell m-\ell b}; q^{\ell^2}\right]_{\infty}} \\
& \times \sum_{\substack{k=0, \\ k \not\equiv b+\frac{1}{2} \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k-b-\frac{1}{2})(k-b+\frac{1}{2}) \frac{\left[q^{\frac{\ell(\ell-1)}{2}+\ell m+\ell k-\ell b}, q^{\ell k-\ell m}; q^{\ell^2}\right]_{\infty}}{\left[q^{\frac{\ell(\ell-1)}{2}-\ell b+\ell k}; q^{\ell^2}\right]_{\infty}} \pmod{\ell}.
\end{aligned}$$

To finish the proof, we must verify that

$$\sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \equiv 2b(-1)^{\frac{\ell-1}{2}} q^{\frac{1-\ell^2}{8}} \frac{(q; q)_{\infty}^3}{(q^{\ell^2}; q^{\ell^2})_{\infty}} \pmod{\ell}.$$

By  $k \mapsto \ell - k$ , we see that

$$\sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \equiv - \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (c-k)(c-k+1) \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \pmod{\ell},$$

and so

$$\begin{aligned}
& \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (k+c)(k+c+1) \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \\
& \equiv \frac{1}{2} \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} ((k+c)(k+c+1) - (c-k)(c-k+1)) \\
& \equiv -2b \sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} k \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \pmod{\ell}.
\end{aligned}$$

So we need to show

$$\sum_{k=1}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} k \left[q^{\ell k}; q^{\ell^2}\right]_{\infty} \equiv (-1)^{\frac{\ell+1}{2}} q^{\frac{1-\ell^2}{8}} \frac{(q; q)_{\infty}^3}{(q^{\ell^2}; q^{\ell^2})_{\infty}} \pmod{\ell}. \quad (2.3)$$

This actually follows just from Jacobi's identity for  $(q; q)_{\infty}^3$ , which we write as

$$\begin{aligned}
(q; q)_{\infty}^3 &= \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}} + \sum_{n=0}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}} + \sum_{n=0}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} (n-1) q^{\frac{n(n-1)}{2}} + \sum_{n=-\infty}^0 (-1)^{n+1} n q^{\frac{n(n-1)}{2}} + \sum_{n=0}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} (-1)^{n+1} n q^{\frac{n(n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n n q^{\frac{n(n-1)}{2}} + \sum_{n=0}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} (-1)^{n+1} n q^{\frac{n(n-1)}{2}}.
\end{aligned}$$

We set  $d = \frac{1-\ell}{2}$  and let  $n \mapsto \ell n + k + d$ , for  $k = 0, 1, \dots, \ell - 1$ , so that

$$\begin{aligned}
(q; q)_{\infty}^3 &\equiv \sum_{k=0}^{\ell-1} (-1)^{k+d+1} (k+d) q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{\ell^2 n^2 - \ell n}{2} + \ell d n + \ell k n} \\
&\equiv \sum_{k=0}^{\ell-1} (-1)^{k+d+1} k q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{\ell^2 n^2 - \ell n}{2} + \ell d n + \ell k n}
\end{aligned}$$

$$\begin{aligned}
& + d \sum_{k=0}^{\ell-1} (-1)^{k+d+1} q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{\ell^2 n^2 - \ell n}{2} + \ell d n + \ell k n} \\
& \equiv \sum_{k=0}^{\ell-1} (-1)^{k+d+1} k q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{\ell^2 n^2 - \ell n}{2} + \ell d n + \ell k n} + d \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{\frac{n(n-1)}{2}} \\
& \equiv \sum_{k=0}^{\ell-1} (-1)^{k+d+1} k q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{\ell^2 n^2 - \ell n}{2} + \ell d n + \ell k n} \\
& \equiv \sum_{k=1}^{\ell-1} (-1)^{k+d+1} k q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \sum_{n=-\infty}^{\infty} (-1)^n q^{\ell k n - \ell^2 n} q^{\frac{\ell^2 n(n+1)}{2}} \\
& \equiv \sum_{k=1}^{\ell-1} (-1)^{k+d+1} k q^{\frac{d(d-1)}{2} + \frac{k(k-1)}{2} + dk} \left( q^{\ell^2}; q^{\ell^2} \right)_{\infty} \left[ q^{\ell k}; q^{\ell^2} \right]_{\infty} \\
& \equiv (-1)^{\frac{1+\ell}{2}} q^{\frac{\ell^2-1}{8}} \sum_{k=1}^{\ell-1} (-1)^k k q^{\frac{k(k-\ell)}{2}} \left( q^{\ell^2}; q^{\ell^2} \right)_{\infty} \left[ q^{\ell k}; q^{\ell^2} \right]_{\infty} \pmod{\ell}. \tag{2.4}
\end{aligned}$$

This immediately implies (2.3) and we have proved the Lemma. A similar congruence for  $(q; q)_{\infty}^3$  can be found in Lemma 3 of [3].  $\square$

While it is not true that  $S_{\ell}(\ell - b - 1) \equiv \pm S_{\ell}(b) \pmod{\ell}$ , they are closely related.

**Lemma 2.2.** *For  $\ell > 1$  odd and  $b$  any integer,*

$$\begin{aligned}
S_{\ell}(\ell - b - 1) & \equiv \frac{2(b+1)(-1)^{b+1} q^{\ell m - \frac{b(b+1)}{2}} (q; q)_{\infty}^3}{(q^{\ell^2}; q^{\ell^2})_{\infty} [q^{\ell m}; q^{\ell^2}]_{\infty}} T\left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2\right) \\
& - (-1)^{\frac{\ell+1}{2} + b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \frac{\left( q^{\ell^2}; q^{\ell^2} \right)_{\infty}^2}{\left[ q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}; q^{\ell^2} \right]_{\infty}} \\
& \times \sum_{\substack{k=0, \\ k \not\equiv b + \frac{1}{2} \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (-k + b + \frac{1}{2})(-k + b + \frac{3}{2}) \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}, q^{\ell k - \ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2} \right]_{\infty}} \\
& \pmod{\ell},
\end{aligned}$$

where  $m$  is any integer such that  $1 \leq m \leq \ell - 1$  and  $m \not\equiv b + \frac{1}{2} \pmod{\ell}$ .

*Proof.* In Lemma we use  $b \mapsto \ell - b - 1$  and  $m \mapsto \ell - m$ . We note that  $m \equiv b + \frac{1}{2} \pmod{\ell}$  if and only if  $\ell - m \equiv \ell - b - 1 + \frac{1}{2} \pmod{\ell}$ . We note that

$$\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1) = \ell^2 - \left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b\right),$$

so that

$$\begin{aligned}
& q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2}} T\left(\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1), \ell(\ell-m), \ell^2\right) \\
& = q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2}} T\left(\ell^2 - \left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b\right), \ell^2 - \ell m, \ell^2\right) \\
& = q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2} - \ell^2 + \frac{\ell(\ell-1)}{2} + 2\ell m - \ell b} T\left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2\right) \\
& = q^{\ell m - \frac{b(b+1)}{2}} T\left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2\right),
\end{aligned}$$

by (2.2). In the sum over  $k$ , we use  $k \mapsto \ell - k$  for  $k$  from 1 to  $\ell - 1$ . We note that

$$\left[ q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1)}; q^{\ell^2} \right]_{\infty} = \left[ q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}; q^{\ell^2} \right]_{\infty},$$

$$\begin{aligned}
\left[ q^{\ell(\ell-k)-\ell(\ell-m)}; q^{\ell^2} \right]_{\infty} &= \left[ q^{\ell m - \ell k}; q^{\ell^2} \right]_{\infty} = -q^{\ell m - \ell k} \left[ q^{\ell k - \ell m}; q^{\ell^2} \right]_{\infty}, \\
\left[ q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) + \ell(\ell-k) - \ell(\ell-b-1)}; q^{\ell^2} \right]_{\infty} &= \left[ q^{2\ell^2 - (\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b)}; q^{\ell^2} \right]_{\infty} \\
&= \left[ q^{-\ell^2 + \frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}; q^{\ell^2} \right]_{\infty} \\
&= -q^{-\ell^2 + \frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b} \left[ q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}; q^{\ell^2} \right]_{\infty}, \\
\left[ q^{\frac{\ell(\ell-1)}{2} - \ell(\ell-b-1) + \ell(\ell-k)}; q^{\ell^2} \right]_{\infty} &= \left[ q^{\ell^2 - (\frac{\ell(\ell-1)}{2} - \ell b + \ell k)}; q^{\ell^2} \right]_{\infty} = \left[ q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2} \right]_{\infty}.
\end{aligned}$$

Thus

$$\begin{aligned}
&q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2}} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) + \ell(\ell-k) - \ell(\ell-b-1)}, q^{\ell(\ell-k) - \ell(\ell-m)}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell(\ell-m)}, q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1)}, q^{\frac{\ell(\ell-1)}{2} - \ell(\ell-b-1) + \ell(\ell-k)}; q^{\ell^2} \right]_{\infty}} \\
&= q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2} - \ell^2 + \frac{\ell(\ell-1)}{2} + 2\ell m - \ell b} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}, q^{\ell k - \ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2} \right]_{\infty}} \\
&= q^{\ell m - \frac{b(b+1)}{2}} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}, q^{\ell k - \ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2} \right]_{\infty}}.
\end{aligned}$$

For the  $k = 0$  term we have

$$\begin{aligned}
&q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2}} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1)}, q^{-\ell(\ell-m)}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell(\ell-m)}, q^{\frac{\ell(\ell-1)}{2} + \ell(\ell-m) - \ell(\ell-b-1)}, q^{\frac{\ell(\ell-1)}{2} - \ell(\ell-b-1)}; q^{\ell^2} \right]_{\infty}} \\
&= -q^{\ell(\ell-m) - \frac{(\ell-b-1)(\ell-b)}{2} - \ell^2 + 2\ell m + \frac{\ell(\ell-1)}{2} - \ell b} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{-\ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{\frac{\ell(\ell-1)}{2} - \ell b}; q^{\ell^2} \right]_{\infty}} \\
&= -q^{\ell m - \frac{b(b+1)}{2}} \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{-\ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}, q^{\frac{\ell(\ell-1)}{2} - \ell b}; q^{\ell^2} \right]_{\infty}}.
\end{aligned}$$

With these identities we have that

$$\begin{aligned}
S_{\ell}(\ell - b - 1) &\equiv \frac{2(b+1)(-1)^{b+1} q^{\ell m - \frac{b(b+1)}{2}} (q; q)_{\infty}^3 T\left(\frac{\ell(\ell-1)}{2} + \ell m - \ell b, \ell m, \ell^2\right)}{(q^{\ell^2}; q^{\ell^2})_{\infty} [q^{\ell m}; q^{\ell^2}]_{\infty}} \\
&\quad - (-1)^{\frac{\ell+1}{2} + b} q^{\frac{\ell^2-1}{8} - \frac{b(b+1)}{2} + \ell m} \frac{(q^{\ell^2}; q^{\ell^2})_{\infty}^2}{[q^{\ell m}, q^{\frac{\ell(\ell-1)}{2} + \ell m - \ell b}; q^{\ell^2}]_{\infty}} \\
&\quad \times \sum_{\substack{k=0, \\ k \not\equiv b + \frac{1}{2} \pmod{\ell}}}^{\ell-1} (-1)^k q^{\frac{k(k-\ell)}{2}} (-k + b + \frac{1}{2})(-k + b + \frac{3}{2}) \frac{\left[ q^{\frac{\ell(\ell-1)}{2} + \ell m + \ell k - \ell b}, q^{\ell k - \ell m}; q^{\ell^2} \right]_{\infty}}{\left[ q^{\frac{\ell(\ell-1)}{2} - \ell b + \ell k}; q^{\ell^2} \right]_{\infty}} \\
&\quad \pmod{\ell}.
\end{aligned}$$

□

While Lemmas 2.1 and 2.2 may appear complicated, their use is quite simple. We are trying to determine formulas modulo  $\ell$  for the  $\ell$ -dissections of  $U(q)$  and  $V(q)$ . Since

$$\frac{1}{(1 - q^n)^2} = \sum_{m=0}^{\infty} (m+1) q^{nm} \equiv \sum_{k=0}^{\ell-1} (k+1) q^{nk} \sum_{m=0}^{\infty} q^{\ell m n} \equiv \sum_{k=0}^{\ell-1} \frac{(k+1) q^{nk}}{1 - q^{\ell n}} \equiv \sum_{k=0}^{\ell-2} \frac{(k+1) q^{nk}}{1 - q^{\ell n}} \pmod{\ell},$$

we have that

$$U(q) \equiv \frac{-1}{2(q; q)_\infty^3} \sum_{b=0}^{\ell-2} (b+1)S_\ell(b) \pmod{\ell},$$

$$V(q) \equiv \frac{-1}{2(q; q)_\infty^3} \sum_{b=0}^{\ell-2} (b+1)S_\ell(b+1) \pmod{\ell}.$$

For  $\ell \geq 5$ , we write these sums as

$$\begin{aligned} U(q) &\equiv \frac{-1}{2(q; q)_\infty^3} \sum_{b=0}^{\ell-2} (b+1)S_\ell(b) \\ &\equiv \frac{-1}{2(q; q)_\infty^3} \left( S_\ell(0) + \sum_{b=1}^{\frac{\ell-3}{2}} (b+1)S_\ell(b) + \frac{(\ell+1)}{2} S_\ell\left(\frac{\ell-1}{2}\right) + \sum_{b=\frac{\ell+1}{2}}^{\ell-2} (b+1)S_\ell(b) \right) \\ &\equiv \frac{-1}{2(q; q)_\infty^3} \left( S_\ell(0) + \sum_{b=1}^{\frac{\ell-3}{2}} (b+1)S_\ell(b) + \frac{1}{2} S_\ell\left(\frac{\ell-1}{2}\right) + \sum_{b=1}^{\frac{\ell-3}{2}} (\ell-b)S_\ell(\ell-1-b) \right) \\ &\equiv \frac{-1}{2(q; q)_\infty^3} \left( S_\ell(0) + \frac{1}{2} S_\ell\left(\frac{\ell-1}{2}\right) + \sum_{b=1}^{\frac{\ell-3}{2}} (b+1)S_\ell(b) - bS_\ell(\ell-1-b) \right) \pmod{\ell}, \end{aligned} \quad (2.5)$$

$$V(q) \equiv \frac{-1}{2(q; q)_\infty^3} \left( -\frac{1}{2} S_\ell\left(\frac{\ell-1}{2}\right) - S_\ell(\ell-1) + \sum_{b=0}^{\frac{\ell-5}{2}} (b+1)S_\ell(b+1) - (b+2)S_\ell(\ell-2-b) \right) \pmod{\ell}. \quad (2.6)$$

Lemmas 2.1 and 2.2 give us congruent expressions for  $U(q)$  and  $V(q)$  that consist of at most  $\frac{\ell+1}{2}$  of the  $T(z, w, q)$  series that each contribute to only one of the  $\ell$ -terms of the dissection and infinite products which we can handle separately from the  $T(z, w, q)$  series. That is to say, Lemmas 2.1 and 2.2 turn the proof of Theorem 1.2 into a verification of congruences and identities between infinite products. We now proceed with the proof of Theorem 1.2.

### 3. PROOF OF THEOREM 1.2.

*Proof of (1.1) and (1.2).* We recall that here  $\ell = 3$  and  $P(a) = [q^{3a}; q^9]_\infty$ . We have that

$$U(q) \equiv \frac{-1}{2E(1)^3} (S_3(0) + 2S_3(1)) \pmod{3},$$

$$V(q) \equiv \frac{-1}{2E(1)^3} (S_3(1) + 2S_3(2)) \pmod{3}.$$

We expand  $S_3(0)$  and  $S_3(1)$  with Lemma 2.1 with  $m = 1$  and  $S_3(2)$  with Lemma 2.2 with  $m = 1$ . After elementary simplifications we have that

$$\begin{aligned} S_3(0) + 2S_3(1) &\equiv \frac{qE(9)^2}{P(1)} + \frac{2q^2E(1)^3T(3, 3, 9)}{E(9)P(1)} \pmod{3}, \\ S_3(1) + 2S_3(2) &\equiv \frac{2q^3E(1)^3T(6, 3, 9)}{E(9)P(1)} + \frac{q^2E(1)^3T(3, 3, 9)}{E(9)P(1)} \pmod{3}. \end{aligned}$$

But  $E(1)^3 = (q; q)_\infty^3 \equiv (q^3; q^3)_\infty = E(3) \pmod{3}$ , and so we see that

$$\begin{aligned} U(q) &\equiv \frac{qE(9)^2}{E(3)P(1)} + \frac{2q^2T(3, 3, 9)}{E(9)P(1)} \pmod{3}, \\ V(q) &\equiv \frac{2q^3T(6, 3, 9)}{E(9)P(1)} + \frac{q^2T(3, 3, 9)}{E(9)P(1)} \pmod{3}, \end{aligned}$$

which are (1.1) and (1.2), respectively. □

*Proof of (1.3).* We recall here  $\ell = 5$  and  $P(a) = [q^{5a}; q^{25}]_\infty$ . Expanding (2.5), when  $\ell = 5$ , with Lemmas 2.1 and 2.2 (with  $m = 1$  in each application), yields that

$$E(1)^3 U(q) \equiv \frac{2q^3 E(25)^2}{P(2)} + \frac{qE(25)^2 P(2)}{P(1)^2} + \frac{4q^2 E(1)^3 T(5, 5, 25)}{E(25)P(1)} + \frac{3q^2 E(25)^2}{P(1)} + \frac{4q^4 E(1)^3 T(10, 5, 25)}{E(25)P(1)} \pmod{5} \quad (3.1)$$

Here we note that

$$\frac{1}{E(1)^3} = \frac{1}{(q; q)_\infty^3} \equiv \frac{(q; q)_\infty^2}{(q^5; q^5)_\infty} = \frac{E(1)^2}{E(5)} \pmod{5}.$$

There are various ways to find the 5-dissection of  $E(1)$ . We use [4, Entry 12(v)], which is

$$E(1) = E(25) \left( \frac{P(2)}{P(1)} - q - \frac{q^2 P(1)}{P(2)} \right). \quad (3.2)$$

Squaring (3.2) gives that

$$E(1)^2 = E(25)^2 \left( \frac{P(2)^2}{P(1)^2} - \frac{2qP(2)}{P(1)} - q^2 + \frac{2q^3 P(1)}{P(2)} + \frac{q^4 P(1)^2}{P(2)^2} \right). \quad (3.3)$$

A direct calculation then gives that

$$\begin{aligned} E(1)^2 &\equiv \frac{2q^3 E(25)^2}{P(2)} + \frac{qE(25)^2 P(2)}{P(1)^2} + \frac{3q^2 E(25)^2}{P(1)} \\ &\equiv E(25)^4 \left( \frac{qP(2)^3}{P(1)^4} + \frac{2q^6 P(1)}{P(2)^2} + \frac{q^2 P(2)^2}{P(1)^3} + \frac{2q^7 P(1)^2}{P(2)^3} \right) \pmod{5}. \end{aligned}$$

Thus multiplying (3.1) by (3.3) and reducing modulo 5 yields

$$\begin{aligned} U(q) &\equiv \frac{qE(25)^4 P(2)^3}{E(5)P(1)^4} + \frac{2q^6 E(25)^4 P(1)}{E(5)P(2)^2} + \frac{4q^2 T(5, 5, 25)}{E(25)P(1)} + \frac{q^2 E(25)^4 P(2)^2}{E(5)P(1)^3} + \frac{2q^7 E(25)^4 P(1)^2}{E(5)P(2)^3} \\ &\quad + \frac{4q^4 T(10, 5, 25)}{E(25)P(1)} \pmod{5}. \end{aligned}$$

We must reduce the products to complete the proof of (1.3). For this we note that cubing (3.2) gives

$$E(1)^3 = E(25)^3 \left( \frac{P(2)^3}{P(1)^3} - \frac{3qP(2)^2}{P(1)^2} - \frac{q^6 P(1)^3}{P(2)^3} + 5q^3 - \frac{3q^5 P(1)^2}{P(2)^2} \right),$$

whereas the  $\ell = 5$  case of (2.4) gives that

$$E(1)^3 \equiv E(25)(2qP(1) + P(2)) \pmod{5}.$$

By comparing the coefficients of  $q$  we have that

$$E(25)^3 \left( \frac{2qP(2)^2}{P(1)^2} + \frac{4q^6 P(1)^3}{P(2)^3} \right) \equiv 2qE(25)P(1) \pmod{5},$$

so that

$$\frac{P(2)^2}{P(1)^3} + \frac{2q^5 P(1)^2}{P(2)^3} \equiv \frac{1}{E(25)^2} \pmod{5}. \quad (3.4)$$

Thus

$$\begin{aligned} \frac{qE(25)^4 P(2)^3}{E(5)P(1)^4} + \frac{2q^6 E(25)^4 P(1)}{E(5)P(2)^2} &\equiv \frac{qE(25)^2 P(2)}{E(5)P(1)} \pmod{5}, \\ \frac{q^2 E(25)^4 P(2)^2}{E(5)P(1)^3} + \frac{2q^7 E(25)^4 P(1)^2}{E(5)P(2)^3} &\equiv \frac{q^2 E(25)^2}{E(5)} \pmod{5}, \end{aligned}$$

which finishes the proof of (1.3). □

*Proof of (1.4).* We recall that here  $\ell = 5$  and  $P(a) = [q^{5a}; q^{25}]_\infty$ . Expanding (2.6), when  $\ell = 5$ , with Lemmas 2.1 and 2.2 (with  $m = 1$  in each application and viewing  $S_\ell(\ell - 1)$  as  $S_\ell(\ell - 0 - 1)$ ), yields that

$$E(1)^3 V(q) \equiv \frac{4q^5 E(1)^3 T(15, 5, 25)}{E(25)P(1)} + \frac{q^2 E(1)^3 T(5, 5, 25)}{E(25)P(1)} + \frac{4q^3 E(25)}{P(2)} + \frac{3q^4 E(25)^2 P(1)}{P(2)^2} \pmod{5}.$$

Again we use that  $\frac{1}{(q; q)_\infty^3} \equiv \frac{(q; q)_\infty^2}{(q^5; q^5)_\infty} \pmod{5}$  and with (3.3) we have that

$$E(1)^2 \left( \frac{4q^3 E(25)^2}{P(2)} + \frac{3q^4 E(25)^2 P(1)}{P(2)^2} \right) \equiv E(25)^4 \left( \frac{4q^3 P(2)}{P(1)^2} + \frac{3q^8 P(1)^3}{P(2)^4} \right) \pmod{5}.$$

Thus

$$V(q) \equiv \frac{4q^5 T(15, 5, 25)}{E(25)P(1)} + \frac{q^2 T(5, 5, 25)}{E(25)P(1)} + \frac{4q^3 E(25)^4 P(2)}{E(5)P(1)^2} + \frac{3q^8 E(25)^4 P(1)^3}{E(5)P(2)^4} \pmod{5}.$$

However, by (3.4) we see that

$$\frac{4q^3 E(25)^4 P(2)}{E(5)P(1)^2} + \frac{3q^8 E(25)^4 P(1)^3}{E(5)P(2)^4} \equiv \frac{4q^3 E(25)^2 P(1)}{E(5)P(2)} \pmod{5},$$

which finishes the proof of (1.4).  $\square$

*Proof of (1.5).* We recall that here  $\ell = 7$  and so  $P(a) = [q^{7a}; q^{49}]_\infty$ . We expand the  $\ell = 7$  case of (2.5) with Lemmas 2.1 and 2.2, using  $m = 1$  in each application. This gives that

$$\begin{aligned} E(1)^3 U(q) &\equiv \frac{5q^7 E(49)^2 P(2)}{P(3)^2} + \frac{3q E(49)^2 P(2)}{P(1)^2} + \frac{5q E(1)^3 T(7, 7, 49)}{E(49)P(1)} + \frac{6q^8 E(49)^2 P(1)}{P(2)P(3)} + \frac{2q^2 E(49)^2 P(2)^2}{P(1)^2 P(3)} \\ &+ \frac{q^2 E(49)^2 P(3)}{P(1)P(2)} + \frac{2q^4 E(1)^3 T(14, 7, 49)}{E(49)P(1)} + \frac{5q^4 E(49)^2 P(3)}{P(2)^2} + \frac{4q^5 E(49)^2}{P(2)} + \frac{q^6 E(49)^2}{P(3)} \\ &+ \frac{4q^6 E(1)^3 T(21, 7, 49)}{E(49)P(1)} \pmod{7}. \end{aligned} \quad (3.5)$$

We note that

$$\frac{1}{E(1)^3} = \frac{1}{(q; q)_\infty^3} \equiv \frac{(q; q)_\infty^4}{(q^7; q^7)_\infty} = \frac{E(1)^4}{E(7)} \pmod{7}.$$

There are various ways to deduce the 7-dissection of  $E(1)$ , we use [4, Entry 17(v)], which is

$$E(1) = E(49) \left( \frac{P(2)}{P(1)} - \frac{qP(3)}{P(2)} - q^2 + \frac{q^5 P(1)}{P(3)} \right).$$

By (2.4) with  $\ell = 7$ ,

$$E(1)^3 \equiv E(49) (5q^3 P(1) + 4qP(2) + P(3)) \pmod{7}.$$

Thus

$$\begin{aligned} E(1)^4 &\equiv E(49)^2 \left( \frac{P(2)P(3)}{P(1)} + \frac{4qP(2)^2}{P(1)} + \frac{6qP(3)^2}{P(2)} + \frac{5q^8 P(1)^2}{P(3)} + 2q^2 P(3) + q^3 P(2) + \frac{2q^4 P(3)P(1)}{P(2)} \right. \\ &\quad \left. + 3q^5 P(1) + \frac{4q^6 P(1)P(2)}{P(3)} \right) \pmod{7}. \end{aligned}$$

We can simplify this slightly. We use [3, Lemma 4], which is

$$P(b)^2 P(c+d)P(c-d) - P(c)^2 P(b+d)P(b-d) + q^{7(c-d)} P(d)^2 P(b+c)P(b-d) = 0,$$

With  $b = 3$ ,  $c = 2$ , and  $d = 1$  this is

$$P(3)^3 P(1) - P(2)^3 P(3) + q^7 P(1)^3 P(2) = 0, \quad (3.6)$$

which implies that

$$\frac{5q^8 P(1)^2}{P(3)} \equiv \frac{5qP(2)^2}{P(1)} + \frac{2qP(3)^2}{P(2)} \pmod{7}.$$

Thus

$$E(1)^4 \equiv E(49)^2 \left( \frac{P(2)P(3)}{P(1)} + \frac{2qP(2)^2}{P(1)} + \frac{qP(3)^2}{P(2)} + 2q^2P(3) + q^3P(2) + \frac{2q^4P(3)P(1)}{P(2)} + 3q^5P(1) + \frac{4q^6P(1)P(2)}{P(3)} \right) \pmod{7}. \quad (3.7)$$

In fact it would appear that we can do even better. Calculations suggest that

$$\frac{4qP(2)^2}{P(1)} + \frac{6qP(3)^2}{P(2)} + \frac{5q^8P(1)^2}{P(3)} \equiv \frac{3qE(7)^4}{E(49)^2} \pmod{7},$$

however for our applications this form is actually not as usual. Multiplying (3.5) by (3.7) and collecting terms yields that

$$U(q) \equiv \frac{5qT(7, 7, 49)}{E(49)P(1)} + \frac{2q^4T(14, 7, 49)}{E(49)P(1)} + \frac{4q^6T(21, 7, 49)}{E(49)P(1)} + A_7(q) \pmod{7},$$

where

$$\begin{aligned} A_7(q) &= \frac{(q^{49}; q^{49})_\infty^4}{(q^7; q^7)_\infty} (A_{7,0}(q^7) + qA_{7,1}(q^7) + q^2A_{7,2}(q^7) + q^3A_{7,3}(q^7) + q^4A_{7,4}(q^7) + q^5A_{7,5}(q^7) \\ &\quad + q^6A_{7,6}(q^7)), \\ A_{7,0}(q^7) &= \frac{4q^7P(2)^2}{P(1)P(3)} + \frac{3q^7P(3)}{P(2)} + \frac{3q^{14}P(1)^2}{P(3)^2}, \\ A_{7,1}(q^7) &= \frac{3P(2)^2P(3)}{P(1)^3} + \frac{4q^7P(2)^3}{P(1)P(3)^2} + \frac{3q^7P(1)P(3)^2}{P(2)^3}, \\ A_{7,2}(q^7) &= \frac{4P(3)^2}{P(1)^2} + \frac{P(2)^3}{P(1)^3} + \frac{q^7P(1)P(3)}{P(2)^2} + \frac{2q^7P(2)}{P(3)}, \\ A_{7,3}(q^7) &= \frac{3P(2)P(3)}{P(1)^2} + \frac{4P(2)^4}{P(1)^3P(3)} + \frac{P(3)^3}{P(2)^2P(1)} + \frac{4q^7P(1)}{P(2)} + \frac{5q^7P(2)^2}{P(3)^2}, \\ A_{7,4}(q^7) &= 0, \\ A_{7,5}(q^7) &= \frac{2P(2)^3}{P(1)^2P(3)} + \frac{5P(3)^3}{P(2)^3} + \frac{5q^7P(1)^2}{P(2)^2} + \frac{5q^7P(1)P(2)}{P(3)^2}, \\ A_{7,6}(q^7) &= \frac{2P(3)^2}{P(2)^2} + \frac{P(2)}{P(1)} + \frac{4q^7P(1)^2}{P(2)P(3)} + \frac{6q^7P(1)P(2)^2}{P(3)^3}. \end{aligned}$$

Upon verifying that  $A_0 \equiv A_5 \equiv 0 \pmod{7}$ , we find the above is equivalent to (1.5). Multiplying (3.6) by  $\frac{3q^7}{P(1)P(2)P(3)^2}$  yields  $A_0 \equiv 0 \pmod{7}$ . Multiplying (3.6) by  $\frac{5}{P(1)P(2)^3}$  and  $\frac{5}{P(1)^2P(3)^2}$ , gives that

$$\begin{aligned} 5\frac{P(3)^3}{P(2)^3} - 5\frac{P(3)}{P(1)} + 5q^7\frac{P(1)^2}{P(2)^2} &= 0, \\ 5\frac{P(3)}{P(1)} - 5\frac{P(2)^3}{P(1)^2P(3)} + 5q^7\frac{P(1)P(2)}{P(3)^2} &= 0, \end{aligned}$$

which when added together imply  $A_5 \equiv 0 \pmod{7}$ . While we can use (3.6) to reduce  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_6$ , it would appear that the reductions are not significant and so we leave these terms as they are.  $\square$

*Proof.* Proof of (1.6). We recall that here  $\ell = 7$  and  $P(a) = [q^{7a}; q^{49}]_\infty$ . We expand the  $\ell = 7$  case of (2.6) with Lemmas 2.1 and 2.2, with  $m = 1$  in each case, to find that

$$\begin{aligned} E(1)^3V(q) &\equiv \frac{6q^7E(1)^3T(28, 7, 49)}{E(49)P(1)} + \frac{q^7E(49)^2P(2)}{P(3)^2} + \frac{2qE(1)^3T(7, 7, 49)}{E(49)P(1)} + \frac{5qE(49)^2P(2)}{P(1)^2} \\ &\quad + \frac{4q^8E(49)^2P(1)}{P(2)P(3)} + \frac{6q^2E(49)^2P(3)}{P(1)P(2)} + \frac{q^2E(49)^2P(2)^2}{P(1)^2P(3)} + \frac{q^3E(49)^2}{P(1)} + \frac{q^4E(1)^3T(14, 7, 49)}{E(49)P(1)} \end{aligned}$$



$$\begin{aligned}
& + \frac{3q^4 E(49)^2 P(2)}{P(1)P(3)} + \frac{3q^4 E(49)^2 P(3)}{P(2)^2} + \frac{2q^5 E(49)^2}{P(2)} + \frac{5q^6 E(1)^3 T(21, 7, 49)}{E(49)P(1)} + \frac{3q^6 E(49)^2}{P(3)} \\
& \pmod{7}.
\end{aligned} \tag{3.8}$$

We multiply (3.8) by (3.7) to find that

$$V(q) \equiv \frac{6q^7 T(28, 7, 49)}{E(49)P(1)} + \frac{2qT(7, 7, 49)}{E(49)P(1)} + \frac{5q^6 T(21, 7, 49)}{E(49)P(1)} + \frac{q^4 T(14, 7, 49)}{E(49)P(1)} + B_7(q) \pmod{7},$$

where

$$\begin{aligned}
B_7(q) &= \frac{E(49)^4}{E(7)} \left( B_{7,0}(q^7) + qB_{7,1}(q^7) + q^2 B_{7,2}(q^7) + q^3 B_{7,3}(q^7) + q^4 B_{7,4}(q^7) + q^5 B_{7,5}(q^7) + q^6 B_{7,6}(q^7) \right), \\
B_{7,0}(q^7) &= \frac{2q^7 P(3)}{P(2)} + \frac{5P(2)^2 q^7}{P(3)P(1)} + \frac{2q^{14} P(1)^2}{P(3)^2}, \\
B_{7,1}(q^7) &= \frac{5P(2)^2 P(3)}{P(1)^3} + \frac{6q^7 P(2)^3}{P(1)P(3)^2} + \frac{6q^7 P(1)P(3)^2}{P(2)^3} + 4q^7, \\
B_{7,2}(q^7) &= \frac{4P(3)^2}{P(1)^2} + \frac{4P(2)^3}{P(1)^3} + \frac{5q^7 P(2)}{P(3)} + \frac{3q^7 P(3)P(1)}{P(2)^2}, \\
B_{7,3}(q^7) &= \frac{2P(2)^4}{P(1)^3 P(3)} + \frac{6P(3)^3}{P(2)^2 P(1)} + \frac{3P(2)P(3)}{P(1)^2} + \frac{4q^7 P(1)}{P(2)} + \frac{6q^7 P(2)^2}{P(3)^2}, \\
B_{7,4}(q^7) &= \frac{5P(2)^2}{P(1)^2} + \frac{2P(3)^2}{P(2)P(1)} + \frac{2q^7 P(1)}{P(3)}, \\
B_{7,5}(q^7) &= \frac{P(3)}{P(1)} + \frac{3P(3)^3}{P(2)^3} + \frac{q^7 P(1)^2}{P(2)^2} + \frac{q^7 P(1)P(2)}{P(3)^2}, \\
B_{7,6}(q^7) &= \frac{3P(2)}{P(1)} + \frac{6P(3)^2}{P(2)^2} + \frac{5q^7 P(1)^2}{P(2)P(3)} + \frac{4q^7 P(1)P(2)^2}{P(3)^3}.
\end{aligned}$$

We use a computer algebra system (in particular we used Maple) to reduce the  $B_{7,i}$  with (3.6). By doing so we find each  $B_{7,i}$  reduces modulo 7 to the corresponding products in (1.6), which completes the proof of (1.6).  $\square$

*Proof of (1.7).* We recall here  $\ell = 13$  and  $P(a) = [q^{13a}; q^{169}]_\infty$ . We expand the  $\ell = 13$  case of (2.5) with Lemmas 2.1 and 2.2, using  $m = 1$  in each application, to obtain

$$\begin{aligned}
E(1)^3 U(q) &\equiv \frac{12q^{13} E(169)^2 P(2)}{P(1)P(4)} + \frac{12q^{13} E(169)^2 P(2)P(4)}{P(1)P(3)P(6)} + \frac{9q^{13} E(169)^2 P(5)}{P(3)P(4)} + \frac{8q^{13} E(169)^2 P(3)P(4)}{P(1)P(5)^2} + \frac{9q E(169)^2 P(5)}{P(1)P(3)} \\
&+ \frac{q E(169)^2 P(2)}{P(1)^2} + \frac{3q E(169)^2 P(3)P(6)}{P(1)P(2)P(5)} + \frac{12q^{14} E(169)^2 P(4)}{P(2)P(6)} + \frac{q^2 E(169)^2 P(3)}{P(1)P(2)} + \frac{2q^2 E(169)^2 P(2)P(5)}{P(1)^2 P(6)} \\
&+ \frac{7q^{15} E(169)^2}{P(3)} + \frac{4q^{15} E(169)^2 P(3)^2}{P(1)P(4)P(6)} + \frac{2q^{15} E(169)^2 P(2)}{P(1)P(5)} + \frac{12q^3 E(1)^3 T(39, 13, 169)}{E(169)P(1)} \\
&+ \frac{3q^3 E(169)^2 P(2)P(4)}{P(1)^2 P(5)} + \frac{10q^3 E(169)^2 P(3)P(5)}{P(1)P(2)P(6)} + \frac{6q^3 E(169)^2 P(4)}{P(1)P(3)} + \frac{8q^3 E(169)^2 P(6)}{P(1)P(4)} + \frac{7q^{16} E(169)^2 P(2)}{P(1)P(6)} \\
&+ \frac{12q^4 E(169)^2 P(5)}{P(1)P(4)} + \frac{5q^4 E(169)^2 P(4)}{P(2)^2} + \frac{4q^4 E(169)^2 P(2)P(3)}{P(1)^2 P(4)} + \frac{10q^4 E(169)^2 P(4)P(5)}{P(1)P(3)P(6)} + \frac{q^{17} E(169)^2 P(6)}{P(4)P(5)} \\
&+ \frac{10q^{-8} E(1)^3 T(13, 13, 169)}{E(169)P(1)} + \frac{3q^{-8} E(169)^2 P(5)}{P(1)^2} + \frac{6q^5 E(169)^2 P(3)P(4)}{P(1)P(2)P(6)} + \frac{5q^5 E(169)^2 P(2)^2}{P(1)^2 P(3)} \\
&+ \frac{9q^{18} E(169)^2 P(2)P(4)}{P(1)P(5)P(6)} + \frac{10q^{18} E(169)^2}{P(4)} + \frac{4q^{-7} E(169)^2 P(4)P(6)}{P(1)^2 P(5)} + \frac{2q^6 E(169)^2}{P(1)} + \frac{9q^6 E(169)^2 P(5)}{P(2)P(3)} \\
&+ \frac{6q^6 E(169)^2 P(4)^2}{P(1)P(3)P(6)} + \frac{7q^{19} E(169)^2 P(2)P(3)}{P(1)P(4)P(6)} + \frac{11q^7 E(1)^3 T(52, 13, 169)}{E(169)P(1)} + \frac{2q^7 E(169)^2 P(3)^2}{P(1)P(2)P(5)} \\
&+ \frac{11q^7 E(169)^2 P(3)P(5)}{P(1)P(4)^2} + \frac{4q^7 E(169)^2 P(2)P(6)}{P(1)P(3)P(4)} + \frac{3q^{20} E(169)^2}{P(5)} + \frac{2q^{-5} E(169)^2 P(3)P(5)}{P(1)^2 P(4)} + \frac{q^8 E(169)^2 P(4)}{P(2)P(3)} \\
&+ \frac{7q^8 E(169)^2 P(4)}{P(1)P(5)} + \frac{5q^{21} E(169)^2}{P(6)} + \frac{2q^9 E(169)^2 P(4)}{P(1)P(6)} + \frac{11q^9 E(169)^2 P(3)}{P(1)P(4)} + \frac{2q^9 E(169)^2 P(2)P(6)}{P(1)P(3)P(5)} \\
&+ \frac{q^{22} E(169)^2 P(5)}{P(6)^2} + \frac{q^{10} E(1)^3 T(65, 13, 169)}{E(169)P(1)} + \frac{10q^{10} E(169)^2 P(3)P(5)}{P(1)P(4)P(6)} + \frac{8q^{10} E(169)^2 P(2)}{P(1)P(3)} + \frac{4q^{10} E(169)^2 P(6)}{P(2)P(5)} \\
&+ \frac{10q^{23} E(169)^2 P(4)}{P(5)P(6)} + \frac{8q^{-2} E(1)^3 T(26, 13, 169)}{E(169)P(1)} + \frac{5q^{-2} E(169)^2 P(4)P(6)}{P(1)P(2)P(5)} + \frac{5q^{-2} E(169)^2 P(2)P(4)}{P(1)^2 P(3)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q^{11}E(169)^2P(3)}{P(1)P(5)} + \frac{7q^{11}E(169)^2}{P(2)} + \frac{5q^{-1}E(169)^2P(3)P(6)}{P(1)P(2)P(4)} + \frac{6q^{-1}E(169)^2P(4)}{P(1)P(2)} + \frac{4q^{12}E(1)^3T(78,13,169)}{E(169)P(1)} \\
& + \frac{8q^{12}E(169)^2P(5)}{P(2)P(6)} + \frac{11q^{12}E(169)^2P(2)P(4)}{P(1)P(3)P(5)} \pmod{13}.
\end{aligned} \tag{3.9}$$

We note that

$$\frac{1}{E(1)^3} = \frac{1}{(q; q)_\infty^3} \equiv \frac{(q^{10}; q^{10})_\infty^{10}}{(q^{13}; q^{13})_\infty} = \frac{E(1)^{10}}{E(13)}.$$

By [5, (51)], with a small correction to the  $q^7$  term, we have that

$$\begin{aligned}
E(q)^{10} \equiv & E(169)^2 (P(2)P(4)P(5)P(6) + 3qP(3)^2P(4)P(6) + 9q^2P(1)P(5)P(6)^2 \\
& + 9q^3P(2)P(3)P(5)P(6) + 12q^4P(2)P(3)P(5)^2 + 11q^5P(2)P(3)P(4)P(6) + 6q^5P(1)P(4)P(5)P(6) \\
& + 5q^{18}P(1)P(2)P(3)P(5) + q^{32}P(1)^2P(2)P(3) + 9q^{20}P(1)P(2)P(3)P(4) + 4q^8P(1)P(4)^2P(5) \\
& + 10q^9P(2)^2P(4)P(6) + q^{10}P(1)P(3)P(4)P(6) + 10q^{11}P(1)P(3)P(4)P(5) \\
& + 3q^{12}P(1)P(2)P(5)P(6)) \pmod{13}
\end{aligned}$$

As noted in [12] and [5], one can use [13, (LVII2)], to get that

$$\begin{aligned}
0 = & P(a+d)P(a-d)P(b+c)P(b-c) - P(a+c)P(a-c)P(b+d)P(b-d) \\
& + q^{13(b-c)}P(a+b)P(a-b)P(c+d)P(c-d).
\end{aligned} \tag{3.10}$$

Setting  $a = 5$ ,  $b = 3$ ,  $c = 2$ , and  $d = 1$  in (3.10) gives

$$P(6)P(4)P(5)P(1) - P(6)P(3)P(4)P(2) + q^{13}P(5)P(2)P(3)P(1) = 0,$$

so that

$$\begin{aligned}
E(q)^{10} \equiv & E(169)^2 (P(2)P(4)P(5)P(6) + 3qP(3)^2P(4)P(6) + 9q^2P(1)P(5)P(6)^2 \\
& + 9q^3P(2)P(3)P(5)P(6) + 12q^4P(2)P(3)P(5)^2 + 3q^5P(2)P(3)P(4)P(6) + q^5P(1)P(4)P(5)P(6) \\
& + q^{32}P(1)^2P(2)P(3) + 9q^{20}P(1)P(2)P(3)P(4) + 4q^8P(1)P(4)^2P(5) \\
& + 10q^9P(2)^2P(4)P(6) + q^{10}P(1)P(3)P(4)P(6) + 10q^{11}P(1)P(3)P(4)P(5) \\
& + 3q^{12}P(1)P(2)P(5)P(6)) \pmod{13}
\end{aligned} \tag{3.11}$$

Again it would appear we could do even perhaps, calculations suggest that

$$11q^5P(2)P(3)P(4)P(6) + 6q^5P(1)P(4)P(5)P(6) + 5q^{18}P(1)P(2)P(3)P(5) \equiv \frac{E(13)^{10}}{E(169)^2},$$

however this form is not as useful for our calculations. We multiply (3.9) by (3.11), collect terms, and reduce modulo 13, to find that

$$\begin{aligned}
U(q) \equiv & \frac{12q^3T(39,13,169)}{E(169)P(1)} + \frac{10q^{-8}T(13,13,169)}{E(169)P(1)} + \frac{11q^7T(52,13,169)}{E(169)P(1)} + \frac{q^{10}T(65,13,169)}{E(169)P(1)} + \frac{8q^{-2}T(26,13,169)}{E(169)P(1)} \\
& + \frac{4q^{12}T(78,13,169)}{E(169)P(1)} + A'_{13}(q) \pmod{13},
\end{aligned}$$

where

$$\begin{aligned}
A'_{13}(q) = & \frac{E(169)^4}{E(13)} \left( A'_{13,0}(q^{13}) + qA'_{13,1}(q^{13}) + q^2A'_{13,2}(q^{13}) + q^3A'_{13,3}(q^{13}) + q^4A'_{13,4}(q^{13}) \right. \\
& + q^5A'_{13,5}(q^{13}) + q^6A'_{13,6}(q^{13}) + q^7A'_{13,7}(q^{13}) + q^8A'_{13,8}(q^{13}) + q^9A'_{13,9}(q^{13}) + q^{10}A'_{13,10}(q^{13}) \\
& \left. + q^{11}A'_{13,11}(q^{13}) + q^{12}A'_{13,12}(q^{13}) \right), \\
A'_{13,0}(q^{13}) = & \frac{6P(2)P(5)P(4)P(6)^2}{P(1)P(3)} + \frac{2P(5)^2P(3)P(6)}{P(1)} + \frac{2P(3)^3P(6)^2}{P(1)P(2)} + \frac{12P(5)^2P(4)^2}{P(1)} + \frac{6P(4)P(6)^3}{P(2)} \\
& + \frac{6P(2)P(5)P(3)^2P(6)}{P(1)^2} + \frac{5P(4)^2P(3)^2P(6)}{P(1)P(2)} + \frac{2q^{13}P(2)^2P(5)P(4)^2}{P(1)P(3)} + \frac{q^{13}P(2)P(5)^2P(3)^2}{P(1)P(4)} \\
& + \frac{11q^{13}P(5)P(4)^3P(3)}{P(2)P(6)} + \frac{10q^{13}P(2)P(4)^2P(3)P(6)}{P(1)P(5)} + \frac{5q^{13}P(2)P(5)^2P(4)P(3)}{P(1)P(6)} + \frac{q^{13}P(1)P(5)P(4)^2P(6)}{P(2)P(3)} \\
& + \frac{q^{13}P(2)^3P(3)P(6)}{P(1)^2} + \frac{10q^{13}P(2)^2P(5)P(6)}{P(1)} + \frac{10q^{13}P(2)P(5)^2P(6)}{P(3)} + \frac{5q^{13}P(5)P(4)P(3)^2}{P(2)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{11q^{13}P(1)P(5)P(6)^2}{P(2)} + q^{13}P(4)^2P(6) + 10q^{13}P(3)P(6)^2 + \frac{10q^{26}P(2)^2P(6)^2}{P(5)} + \frac{12q^{26}P(2)P(4)^3}{P(6)} \\
& + 7q^{26}P(2)P(4)P(3) + 11q^{26}P(1)P(5)P(4) + \frac{q^{26}P(5)P(3)^3}{P(6)} + \frac{12q^{26}P(2)P(5)^3P(3)}{P(6)^2} \\
& + \frac{11q^{39}P(2)^2P(3)^2}{P(6)} + \frac{2q^{39}P(1)P(3)^3}{P(5)} + \frac{4q^{39}P(1)P(2)^2P(6)}{P(4)} + \frac{11q^{39}P(1)P(2)P(5)P(3)^2}{P(4)^2} + \frac{3q^{52}P(1)^2P(2)P(3)}{P(5)}, \\
A'_{13,1}(q^{13}) = & \frac{3P(4)^3P(6)}{P(1)} + \frac{9P(4)P(3)P(6)^2}{P(1)} + \frac{2P(5)P(4)P(6)^2}{P(2)} + \frac{9P(2)P(5)^2P(4)P(6)}{P(1)P(3)} + \frac{11P(2)^2P(5)P(4)P(6)}{P(1)^2} \\
& + \frac{6P(5)P(3)P(6)^3}{P(2)P(4)} + \frac{11q^{13}P(2)^4P(4)P(6)}{P(1)^2P(3)} + \frac{6q^{13}P(2)^2P(4)P(6)^2}{P(1)P(5)} + \frac{11q^{13}P(4)^2P(3)^3P(6)}{P(1)P(5)^2} \\
& + \frac{10q^{13}P(1)P(5)^2P(4)^2}{P(2)P(3)} + \frac{9q^{13}P(5)^2P(4)P(3)^2}{P(2)P(6)} + \frac{5q^{13}P(1)P(4)^2P(3)P(6)}{P(2)^2} + \frac{2q^{13}P(2)P(4)^2P(3)}{P(1)} \\
& + \frac{10q^{13}P(2)P(4)P(6)^2}{P(3)} + \frac{4q^{13}P(2)P(3)^2P(6)}{P(1)} + \frac{11q^{13}P(5)P(4)^4}{P(3)P(6)} + \frac{7q^{13}P(1)P(5)^2P(6)}{P(2)} + \frac{11q^{13}P(2)^2P(5)^2}{P(1)} \\
& + \frac{3q^{13}P(2)P(5)^3P(3)^2}{P(1)P(4)P(6)} + 10q^{13}P(5)P(3)P(6) + q^{13}P(5)P(4)^2 + \frac{8q^{26}P(1)P(2)P(5)P(6)}{P(3)} + \frac{12q^{26}P(2)^3P(4)^2}{P(1)P(5)} \\
& + \frac{9q^{26}P(2)P(5)P(3)^2}{P(4)} + \frac{q^{26}P(1)P(5)^2P(4)}{P(6)} + \frac{q^{26}P(1)P(3)P(6)^2}{P(5)} + \frac{5q^{26}P(4)P(3)^3}{P(5)} + 12q^{26}P(2)^2P(6) \\
& + \frac{8q^{39}P(1)P(2)P(4)P(3)}{P(5)} + q^{39}P(1)^2P(4) + \frac{5q^{52}P(1)^2P(2)P(3)}{P(6)}, \\
A'_{13,2}(q^{13}) = & \frac{9P(4)P(3)^3P(6)^2}{P(1)^2P(5)} + \frac{P(2)^2P(4)^2P(6)^2}{P(1)^2P(5)} + \frac{3P(2)P(4)P(3)^2P(6)}{P(1)^2} + \frac{2P(5)P(4)P(3)P(6)}{P(1)} + \frac{6P(5)P(3)^2P(6)^2}{P(1)P(4)} \\
& + \frac{10P(2)^2P(5)^2P(4)}{P(1)^2} + \frac{3q^{13}P(1)P(5)^2P(6)^2}{P(4)P(3)} + \frac{11q^{13}P(1)P(5)P(4)^2P(3)}{P(2)^2} + \frac{9q^{13}P(5)^2P(4)^2}{P(6)} \\
& + \frac{11q^{13}P(4)^2P(3)^2}{P(2)} + \frac{8q^{13}P(2)^2P(4)^3}{P(1)P(3)} + \frac{3q^{13}P(2)^2P(4)P(6)}{P(1)} + \frac{2q^{13}P(2)P(5)P(6)^2}{P(4)} + \frac{6q^{13}P(4)P(3)P(6)^2}{P(5)} \\
& + \frac{4q^{13}P(1)P(4)P(6)^2}{P(2)} + 9q^{13}P(5)^2P(3) + \frac{11q^{26}P(2)P(4)^2P(3)}{P(5)} + \frac{5q^{26}P(2)^3P(3)}{P(1)} + 7q^{26}P(1)P(3)P(6) \\
& + 8q^{26}P(2)^2P(5) + 5q^{26}P(1)P(4)^2 + \frac{8q^{39}P(1)P(2)P(4)P(3)}{P(6)} + \frac{2q^{39}P(1)P(2)^2P(6)}{P(5)} + \frac{11q^{39}P(1)P(2)P(3)^2}{P(4)} \\
& + \frac{q^{52}P(1)^2P(2)P(5)P(3)}{P(6)^2}, \\
A'_{13,3}(q^{13}) = & \frac{6P(2)P(5)P(4)P(3)^2}{P(1)^2} + \frac{6P(4)^2P(3)P(6)^2}{P(1)P(5)} + \frac{8P(5)^2P(3)^2P(6)}{P(1)P(4)} + \frac{3P(4)P(3)^3P(6)}{P(1)P(2)} + \frac{3P(5)^2P(6)^2}{P(3)} \\
& + \frac{5P(4)^2P(6)^2}{P(2)} + \frac{P(3)P(6)^3}{P(2)} + \frac{11P(2)P(5)P(4)^2P(6)}{P(1)P(3)} + \frac{5P(2)^2P(4)^2P(6)}{P(1)^2} + \frac{4P(2)P(5)P(6)^2}{P(1)} \\
& + \frac{3P(5)^2P(4)P(3)}{P(1)} + \frac{q^{13}P(2)^3P(6)^2}{P(1)P(3)} + \frac{4q^{13}P(1)P(5)P(4)^3}{P(2)P(3)} + \frac{8q^{13}P(5)P(4)^2P(3)^2}{P(2)P(6)} + \frac{9q^{13}P(1)P(5)P(4)P(6)}{P(2)} \\
& + \frac{12q^{13}P(3)^4}{P(1)} + \frac{9q^{13}P(2)^2P(5)P(3)P(6)}{P(1)P(4)} + \frac{10q^{13}P(2)P(4)P(3)^2P(6)}{P(1)P(5)} + \frac{11q^{13}P(2)^2P(5)P(4)}{P(1)} \\
& + \frac{4q^{13}P(2)P(5)^2P(4)}{P(3)} + \frac{5q^{13}P(5)^3P(3)}{P(6)} + 8q^{13}P(4)^3 + 6q^{13}P(4)P(3)P(6) + \frac{9q^{26}P(2)^2P(4)P(6)}{P(5)} \\
& + \frac{4q^{26}P(2)P(4)^2P(3)}{P(6)} + \frac{3q^{26}P(1)P(2)P(6)^2}{P(4)} + \frac{7q^{26}P(1)P(5)P(4)^2}{P(6)} + 9q^{26}P(1)P(5)P(3) + 2q^{26}P(2)P(3)^2 \\
& + \frac{9q^{39}P(1)P(2)P(5)P(4)P(3)}{P(6)^2} + \frac{10q^{39}P(1)P(2)P(5)P(3)^2}{P(4)P(6)} + \frac{4q^{39}P(1)^2P(3)P(6)}{P(5)} + 8q^{39}P(1)P(2)^2 \\
& + \frac{10q^{52}P(1)^2P(2)P(4)P(3)}{P(5)P(6)}, \\
A'_{13,4}(q^{13}) = & \frac{9P(2)P(4)^2P(3)^2P(6)}{P(1)^2P(5)} + \frac{7P(2)^2P(5)P(3)P(6)}{P(1)^2} + \frac{10P(2)P(5)^2P(4)^2}{P(1)P(3)} + \frac{4P(5)P(4)P(3)^3}{P(1)P(2)} + \frac{3P(2)P(5)^2P(6)}{P(1)} \\
& + \frac{11P(4)^2P(3)P(6)}{P(1)} + \frac{11P(5)P(4)^2P(6)}{P(2)} + \frac{P(5)P(3)P(6)^2}{P(2)} + \frac{P(3)^2P(6)^2}{P(1)} + \frac{3q^{13}P(2)P(4)^2P(6)}{P(3)} \\
& + \frac{11q^{13}P(1)P(5)P(6)^2}{P(3)} + \frac{8q^{13}P(5)P(3)^2P(6)}{P(4)} + \frac{7q^{13}P(1)P(5)^2P(4)}{P(2)} + \frac{4q^{13}P(2)P(5)^3}{P(4)} + \frac{3q^{13}P(5)P(4)^3}{P(6)} \\
& + \frac{2q^{13}P(2)^3P(5)P(6)}{P(1)P(3)} + \frac{12q^{13}P(2)^2P(4)^2P(6)}{P(1)P(5)} + \frac{q^{13}P(2)^2P(5)^2P(4)}{P(1)P(6)} + \frac{q^{13}P(2)^2P(5)^2P(3)}{P(1)P(4)} \\
& + \frac{2q^{13}P(4)P(3)^3P(6)}{P(2)P(5)} + 10q^{13}P(2)P(6)^2 + 6q^{13}P(5)P(4)P(3) + \frac{4q^{26}P(1)P(2)P(5)P(6)}{P(4)} \\
& + \frac{5q^{26}P(1)P(4)P(3)P(6)}{P(5)} + \frac{4q^{26}P(2)P(5)P(3)^2}{P(6)} + \frac{4q^{26}P(1)P(5)^2P(4)^2}{P(6)^2} + 11q^{26}P(2)^2P(4) + \frac{q^{39}P(1)P(2)P(3)^2}{P(5)} \\
& + \frac{12q^{39}P(1)P(2)P(4)^2P(3)}{(P(5)P(6))} + 7q^{39}P(1)^2P(3), \\
A'_{13,5}(q^{13}) = & + \frac{3P(2)P(5)^2P(4)P(6)}{q^{13}P(1)^2} + \frac{5P(2)^3P(5)P(4)P(6)}{P(1)^2P(3)} + \frac{4P(2)P(5)^3}{P(1)} + \frac{7P(5)P(6)^3}{P(4)} + \frac{4P(2)^2P(5)^2P(3)}{P(1)^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{12P(2)P(3)^3P(6)}{P(1)^2} + \frac{10P(5)P(4)^2P(3)}{P(1)} + \frac{2P(5)P(4)P(6)^2}{P(3)} + \frac{5P(5)P(3)^2P(6)}{P(1)} + \frac{12P(5)^2P(3)P(6)}{P(2)} \\
& + \frac{2P(4)^2P(3)^2P(6)}{P(2)^2} + \frac{10q^{13}P(2)P(5)P(3)^3}{P(1)P(4)} + \frac{6q^{13}P(2)^2P(4)^2}{P(1)} + \frac{3q^{13}P(3)^2P(6)^2}{P(5)} + \frac{6q^{13}P(5)^2P(3)^2}{P(4)} \\
& + \frac{7q^{13}P(4)P(3)^3}{P(2)} + \frac{8q^{13}P(2)^2P(3)P(6)}{P(1)} + \frac{10q^{13}P(2)P(5)P(4)^2}{P(3)} + \frac{8q^{13}P(4)^2P(3)P(6)}{P(5)} + \frac{2q^{13}P(5)^2P(4)P(3)}{P(6)} \\
& + \frac{10q^{13}P(1)P(5)^2P(6)}{P(3)} + \frac{4q^{13}P(1)P(4)^2P(6)}{P(2)} + \frac{7q^{13}P(2)^3P(4)P(6)^2}{P(1)P(5)P(3)} + \frac{11q^{13}P(2)P(4)^2P(3)^2P(6)}{P(1)P(5)^2} \\
& + \frac{q^{26}P(1)P(4)^3}{P(6)} + \frac{10q^{26}P(2)^2P(5)P(4)}{P(6)} + \frac{8q^{26}P(2)^2P(5)P(3)}{P(4)} + \frac{9q^{26}P(2)P(4)P(3)^2}{P(5)} + \frac{5q^{26}P(1)P(3)^2P(6)}{P(4)} \\
& + \frac{11q^{39}P(1)P(2)^2P(4)}{P(5)} + \frac{8q^{39}P(1)^2P(5)P(3)}{P(6)}, \\
A'_{13,6}(q^{13}) = & \frac{4P(2)P(4)^2P(6)^2}{q^{13}P(1)^2} + \frac{9P(5)P(4)P(3)^2P(6)}{q^{13}P(1)^2} + \frac{4P(2)P(5)P(3)P(6)^2}{P(1)P(4)} + \frac{5P(5)^2P(3)^2}{P(1)} + \frac{4P(5)^2P(6)^2}{P(4)} \\
& + \frac{7P(4)^3P(6)}{P(2)} + \frac{6P(2)^2P(4)P(3)P(6)}{P(1)^2} + \frac{11P(2)^2P(5)^3P(3)}{P(1)^2P(6)} + \frac{6P(2)P(5)P(4)P(6)}{P(1)} + \frac{9P(4)P(3)^2P(6)^2}{P(1)P(5)} \\
& + \frac{6P(1)P(5)P(4)P(6)^2}{P(2)^2} + \frac{4P(5)^2P(4)P(6)}{P(3)} + \frac{5P(4)^2P(3)^3}{P(1)P(2)} + \frac{3P(4)P(3)P(6)^2}{P(2)} + \frac{9q^{13}P(1)P(6)^3}{P(4)} \\
& + \frac{2q^{13}P(2)^3P(4)P(6)}{P(1)P(3)} + \frac{12q^{13}P(2)^2P(5)P(6)^2}{P(4)P(3)} + \frac{q^{13}P(2)P(4)^2P(3)^2}{P(1)P(5)} + \frac{7q^{13}P(2)P(5)^2P(3)P(6)}{P(4)^2} \\
& + \frac{9q^{13}P(2)P(5)^2P(3)^3}{P(1)P(4)P(6)} + \frac{12q^{13}P(2)^2P(5)P(3)}{P(1)} + \frac{3q^{13}P(2)P(4)P(6)^2}{P(5)} + \frac{11q^{13}P(1)P(5)P(4)^2}{P(2)} + 6q^{13}P(2)P(5)^2 \\
& + q^{13}P(3)^2P(6) + 10q^{13}P(4)^2P(3) + \frac{4q^{26}P(2)^2P(4)^2}{P(5)} + 9q^{26}P(1)P(2)P(6) + \frac{6q^{26}P(1)P(5)P(4)P(3)}{P(6)} \\
& + \frac{8q^{39}P(1)P(2)P(4)P(3)^2}{P(5)^2} + \frac{12q^{39}P(1)P(2)^2P(4)}{P(6)} + \frac{12q^{39}P(1)P(2)^2P(3)}{P(4)} + \frac{9q^{39}P(1)^2P(2)P(5)}{P(4)}, \\
A'_{13,7}(q^{13}) = & \frac{12P(4)^2P(3)^2P(6)^2}{q^{13}P(1)^2P(5)} + \frac{P(5)^2P(6)^2}{q^{13}P(1)} + \frac{5P(4)^3P(3)}{P(1)} + \frac{11P(5)P(4)^3}{P(2)} + \frac{8P(2)P(5)^2P(4)}{P(1)} + \frac{11P(4)P(3)^2P(6)}{P(1)} \\
& + \frac{3P(5)^3P(3)^2}{(P(1)P(6))} + \frac{11P(2)^3P(4)^2P(6)}{P(1)^2P(3)} + \frac{3P(2)^2P(5)P(4)P(3)}{P(1)^2} + \frac{10P(2)^2P(5)P(6)^2}{P(1)P(3)} + \frac{11P(2)P(4)^2P(6)^2}{P(1)P(5)} \\
& + \frac{4P(5)P(4)P(3)P(6)}{P(2)} + \frac{10P(2)^2P(5)P(3)^2P(6)}{P(1)^2P(4)} + \frac{5q^{13}P(2)P(4)^3}{P(3)} + \frac{7q^{13}P(2)P(3)^3}{P(1)} + \frac{3q^{13}P(2)^2P(4)P(3)P(6)}{P(1)P(5)} \\
& + \frac{4q^{13}P(1)P(4)P(3)P(6)^2}{P(2)P(5)} + \frac{9q^{13}P(2)P(3)P(6)^2}{P(4)} + \frac{7q^{13}P(5)P(4)^2P(3)}{P(6)} + \frac{12q^{13}P(1)P(5)P(6)^2}{P(4)} \\
& + \frac{6q^{13}P(2)^2P(5)^2P(3)}{(P(1)P(6))} + \frac{10q^{13}P(1)P(5)P(4)P(6)}{P(3)} + \frac{6q^{13}P(1)P(5)^2P(4)^2}{(P(2)P(6))} + 5q^{13}P(2)P(4)P(6) + 7q^{13}P(5)P(3)^2 \\
& + \frac{7q^{26}P(2)P(4)^2P(3)^2}{P(5)^2} + \frac{10q^{26}P(1)P(4)^2P(3)}{P(5)} + \frac{3q^{26}P(1)P(3)^2P(6)}{P(5)} + \frac{4q^{26}P(2)^2P(4)^2}{P(6)} \\
& + \frac{10q^{26}P(1)P(5)^2P(4)P(3)}{P(6)^2} + 5q^{26}P(2)^2P(3) + q^{26}P(1)P(2)P(5) + \frac{12q^{39}P(1)^2P(4)P(3)}{P(6)}, \\
A'_{13,8}(q^{13}) = & \frac{3P(2)P(5)^2P(3)P(6)}{q^{13}P(1)^2} + \frac{10P(4)P(6)^3}{q^{13}P(1)} + \frac{6P(2)^3P(5)P(6)}{P(1)^2} + \frac{2P(2)P(4)^2P(6)}{P(1)} + \frac{8P(2)P(3)P(6)^2}{P(1)} \\
& + \frac{9P(5)P(4)^2P(6)}{P(3)} + \frac{6P(5)P(4)P(3)^2}{P(1)} + \frac{5P(5)^2P(4)P(3)}{P(2)} + \frac{9P(2)^2P(4)^2P(3)P(6)}{P(1)^2P(5)} + \frac{6P(4)P(3)^4P(6)}{P(1)P(2)P(5)} \\
& + \frac{9P(2)^2P(5)^2P(3)^2}{P(1)^2P(4)} + \frac{3P(2)P(5)^3P(4)}{P(1)P(6)} + \frac{P(2)P(5)^3P(3)}{P(1)P(4)} + \frac{7P(5)P(3)^3P(6)}{P(1)P(4)} + \frac{5P(4)^2P(3)P(6)^2}{P(2)P(5)} \\
& + \frac{3P(1)P(5)^2P(6)^2}{P(2)P(3)} + \frac{9q^{13}P(2)P(5)P(4)^3}{(P(3)P(6))} + \frac{3q^{13}P(2)P(5)P(3)P(6)}{P(4)} + \frac{8q^{13}P(1)P(4)P(3)P(6)}{P(2)} \\
& + \frac{7q^{13}P(2)^2P(4)P(3)}{P(1)} + \frac{10q^{13}P(1)P(5)^2P(4)}{P(3)} + \frac{11q^{13}P(4)P(3)^2P(6)}{P(5)} + \frac{6q^{13}P(2)^2P(6)^2}{P(3)} + \frac{6q^{13}P(4)^3P(3)}{P(5)} \\
& + \frac{9q^{13}P(5)^2P(3)^2}{P(6)} + \frac{6q^{13}P(2)^3P(4)^2P(6)}{(P(1)P(5)P(3))} + 8q^{13}P(2)P(5)P(4) + q^{26}P(1)P(3)^2 + \frac{2q^{26}P(2)^2P(5)P(3)}{P(6)} \\
& + \frac{3q^{26}P(1)P(2)P(5)^2}{P(6)} + \frac{2q^{39}P(1)P(2)^2P(3)}{P(5)} + \frac{4q^{39}P(1)P(2)P(3)^3}{(P(4)P(6))} + 7q^{39}P(1)^2P(2), \\
A'_{13,9}(q^{13}) = & \frac{10P(2)P(4)P(3)P(6)^2}{q^{13}P(1)^2} + \frac{10P(2)P(5)^3P(3)}{q^{13}P(1)^2} + \frac{6P(5)P(3)^3P(6)}{q^{13}P(1)^2} + \frac{8P(2)^3P(5)^2}{P(1)^2} + \frac{7P(5)^2P(4)^2}{P(3)} + \frac{10P(3)^2P(6)^2}{P(2)} \\
& + \frac{2P(2)^2P(4)P(6)^2}{(P(1)P(3))} + \frac{4P(2)P(5)P(3)P(6)}{P(1)} + \frac{10P(2)P(5)P(6)^3}{P(4)P(3)} + \frac{8P(4)^2P(3)^2P(6)}{P(1)P(5)} + \frac{7P(5)^2P(4)P(3)^2}{P(1)P(6)} \\
& + \frac{5P(1)P(5)P(4)^2P(6)}{P(2)^2} + \frac{12P(2)^2P(3)^2P(6)}{P(1)^2} + \frac{10P(2)P(5)P(4)^2}{P(1)} + \frac{8P(5)^2P(3)P(6)^2}{P(4)^2} + \frac{8P(4)^2P(3)P(6)}{P(2)} \\
& + 2P(5)^2P(6) + \frac{11q^{13}P(2)^2P(5)P(3)^2}{P(1)P(4)} + \frac{9q^{13}P(1)P(5)P(4)^3}{P(2)P(6)} + \frac{3q^{13}P(2)^3P(6)}{P(1)} + \frac{3q^{13}P(2)^3P(4)^2}{P(1)P(3)} \\
& + \frac{10q^{13}P(2)^2P(5)P(6)}{P(3)} + \frac{11q^{13}P(2)P(4)^2P(6)}{P(5)} + \frac{q^{13}P(2)P(5)^2P(4)}{P(6)} + \frac{3q^{13}P(2)P(3)P(6)^2}{P(5)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{7q^{13}P(2)P(5)^2P(3)}{P(4)} + \frac{2q^{13}P(2)^2P(4)^2P(3)P(6)}{P(1)P(5)^2} + \frac{9q^{13}P(2)^2P(5)P(4)P(3)}{P(1)P(6)} + q^{13}P(1)P(6)^2 + 8q^{13}P(4)P(3)^2 \\
& + \frac{8q^{26}P(2)^2P(4)P(3)}{P(5)} + \frac{10q^{26}P(1)P(5)P(3)^2}{P(6)} + \frac{8q^{26}P(1)P(2)P(3)P(6)}{P(4)} + \frac{10q^{26}P(2)P(3)^3}{P(6)} + 8q^{26}P(1)P(2)P(4) \\
& + \frac{7q^{39}P(1)P(2)^2P(3)}{P(6)}, \\
A'_{13,10}(q^{13}) = & \frac{5P(5)^2P(3)P(6)^2}{q^{13}P(1)P(4)} + \frac{3P(5)^2P(4)P(6)}{q^{13}P(1)} + \frac{5P(2)P(5)P(4)P(3)P(6)}{q^{13}P(1)^2} + \frac{P(2)^2P(5)P(4)P(6)}{P(1)P(3)} \\
& + \frac{10P(2)P(4)P(3)P(6)^2}{P(1)P(5)} + \frac{8P(2)P(5)^2P(3)^2P(6)}{P(1)P(4)^2} + \frac{9P(1)P(5)P(4)P(6)^2}{P(2)P(3)} + \frac{8P(2)P(5)^2P(3)}{P(1)} \\
& + \frac{5P(5)P(4)^2P(3)}{P(2)} + \frac{11P(5)P(3)^2P(6)}{P(2)} + \frac{11P(4)^2P(3)^2}{P(1)} + \frac{12P(3)^3P(6)}{P(1)} + \frac{10P(2)^2P(5)P(6)^2}{P(1)P(4)} \\
& + \frac{2P(2)P(5)^2P(4)^2}{(1)P(6)} + \frac{12P(2)^3P(4)P(6)}{P(1)^2} + 4P(5)^3 + 4P(4)P(6)^2 + \frac{2q^{13}P(2)^2P(4)^2P(3)}{P(1)P(5)} \\
& + \frac{2q^{13}P(1)P(5)^2P(4)P(3)}{P(2)P(6)} + \frac{2q^{13}P(1)P(5)P(4)^2}{P(3)} + \frac{8q^{13}P(4)^2P(3)^2P(6)}{P(5)^2} + \frac{5q^{13}P(5)P(4)P(3)^2}{P(6)} \\
& + \frac{6q^{13}P(2)^2P(5)^2P(3)^2}{P(1)P(4)P(6)} + q^{13}P(2)P(3)P(6) + 7q^{13}P(1)P(5)P(6) + 11q^{13}P(2)P(4)^2 + \frac{10q^{26}P(1)P(2)P(5)P(4)}{P(6)} \\
& + \frac{12q^{26}P(1)P(2)P(5)P(3)}{P(4)} + \frac{11q^{26}P(2)^2P(4)P(3)}{P(6)} + \frac{5q^{26}P(1)^2P(4)P(3)}{P(2)} + \frac{4q^{26}P(2)^2P(3)^2}{P(4)} \\
& + \frac{q^{39}P(1)^2P(2)P(3)P(6)}{P(5)P(4)}, \\
A'_{13,11}(q^{13}) = & \frac{5P(2)^2P(5)P(4)^2P(6)}{q^{13}P(1)^2P(3)} + \frac{12P(2)P(4)^2P(3)P(6)^2}{q^{13}P(1)^2P(5)} + \frac{5P(2)P(5)^2P(3)^2P(6)}{q^{13}P(1)^2P(4)} + \frac{9P(4)^2P(6)^2}{q^{13}P(1)} + \frac{10P(5)P(3)P(6)^2}{P(4)} \\
& + \frac{7P(2)^3P(5)P(4)}{P(1)^2} + \frac{4P(2)P(4)^3}{P(1)} + \frac{5P(2)P(6)^3}{P(3)} + \frac{4P(5)P(4)^3}{P(3)} + \frac{2P(5)P(3)^3}{P(1)} + \frac{9P(2)^2P(5)^2P(6)}{P(1)P(4)} \\
& + \frac{P(2)P(4)P(3)P(6)}{P(1)} + \frac{2P(2)P(5)^3P(3)^2}{P(1)P(4)^2} + \frac{P(5)^2P(4)^2P(3)}{P(2)P(6)} + \frac{2P(4)P(3)^2P(6)^2}{P(2)P(5)} + \frac{9P(1)P(5)^2P(4)P(6)}{P(2)P(3)} \\
& + 5P(5)P(4)P(6) + \frac{12q^{13}P(2)^2P(4)P(6)}{P(3)} + \frac{4q^{13}P(2)P(5)P(4)^2}{P(6)} + \frac{5q^{13}P(1)P(4)^2P(3)}{P(2)} + \frac{6q^{13}P(2)^2P(3)^2}{P(1)} \\
& + \frac{6q^{13}P(4)^2P(3)^2}{P(5)} + \frac{7q^{13}P(2)^3P(4)P(6)}{P(1)P(5)} + 7q^{13}P(2)P(5)P(3) + 6q^{13}P(1)P(5)^2 + \frac{9q^{26}P(1)P(2)P(3)P(6)}{P(5)} \\
& + \frac{6q^{26}P(1)P(4)P(3)^2}{P(6)} + 5q^{26}P(2)^3 + \frac{10q^{39}P(1)^2P(2)P(3)}{P(4)} + \frac{9q^{39}P(1)P(2)^2P(4)P(3)}{P(5)P(6)}, \\
A'_{13,12}(q^{13}) = & \frac{2P(4)^2P(3)^2P(6)^2}{q^{13}P(1)P(2)P(5)} + \frac{6P(5)P(4)^2P(6)}{q^{13}P(1)} + \frac{5P(5)P(3)P(6)^2}{q^{13}P(1)} + \frac{2P(2)P(4)^2P(3)P(6)}{q^{13}P(1)^2} + \frac{11P(2)P(5)^3P(3)^2}{(q^{13}P(1)^2P(4)} \\
& + \frac{11P(2)P(5)P(6)^2}{P(3)} + \frac{P(5)^2P(4)^3}{(P(3)P(6))} + \frac{10P(5)^2P(3)P(6)}{P(4)} + \frac{4P(2)^3P(4)^2P(6)}{P(1)^2P(5)} + \frac{6P(2)^2P(4)^2P(6)}{P(1)P(3)} \\
& + \frac{10P(2)P(5)P(4)P(3)}{P(1)} + \frac{9P(4)P(3)^3P(6)}{P(1)P(5)} + \frac{2P(4)P(3)^2P(6)}{P(2)} + \frac{7P(1)P(5)P(4)^3}{P(2)^2} + \frac{2P(2)P(5)P(3)^2P(6)}{P(1)P(4)} \\
& + \frac{6P(2)^2P(6)^2}{P(1)} + \frac{10P(1)P(6)^3}{P(2)} + 2P(5)^2P(4) + \frac{11q^{13}P(2)^3P(4)}{P(1)} + \frac{2q^{13}P(4)^2P(3)^2}{P(6)} + \frac{10q^{13}P(2)^2P(5)P(4)}{P(3)} \\
& + \frac{10q^{13}P(2)^2P(5)P(6)}{P(4)} + \frac{4q^{13}P(2)P(5)^2P(3)}{P(6)} + \frac{3q^{13}P(1)P(5)P(4)^2P(3)}{P(2)P(6)} + \frac{q^{13}P(1)P(2)P(5)^2P(6)}{P(4)P(3)} + 4q^{13}P(3)^3 \\
& + \frac{6q^{26}P(1)P(2)P(4)^2}{P(6)} + \frac{3q^{26}P(2)^2P(4)^2P(3)}{P(5)P(6)} + q^{26}P(1)P(2)P(3) + 9q^{26}P(1)^2P(5) + \frac{7q^{39}P(1)P(2)^2P(3)^2}{P(4)P(6)}
\end{aligned}$$

To finish the proof of (1.7) we must show that  $A'_{13} \equiv A_{13} \pmod{13}$ . We do this by reducing each  $A'_{13,i}$  modulo 13. For each of the  $A'_{13,i}$  we clear denominators and use a computer algebra system (in particular we used Maple) to reduce according the following rules (with priority of eliminating  $P(6)$ ,  $P(5)$ ,  $P(4)$ ,  $P(3)$ ,  $P(2)$ , and then  $P(1)$ ):

$$\begin{aligned}
P(3)^3P(1) - P(4)P(2)^3 + q^{13}P(5)P(1)^3 &= 0, \\
P(4)^3P(2) - P(5)P(3)^3 + q^{26}P(6)P(1)^3 &= 0, \\
P(5)^3P(1) - P(6)P(3)^3 + q^{13}P(5)P(2)^3 &= 0, \\
P(5)^3P(3) - P(6)P(4)^3 + q^{39}P(4)P(1)^3 &= 0, \\
P(6)^3P(1) - P(4)^3P(3) + q^{13}P(3)^3P(2) &= 0, \\
P(6)^3P(2) - P(5)P(4)^3 + q^{26}P(3)P(2)^3 &= 0, \\
P(6)^3P(3) - P(5)^3P(4) + q^{39}P(2)^3P(1) &= 0, \\
P(6)^3P(4) - P(6)P(5)^3 + q^{52}P(2)P(1)^3 &= 0,
\end{aligned}$$

$$\begin{aligned}
& P(4)^2 P(3) P(1) - P(5) P(3) P(2)^2 + q^{13} P(6) P(2) P(1)^2 = 0, \\
& P(4)^2 P(5) P(1) - P(6) P(2) P(3)^2 + q^{13} P(6) P(1) P(2)^2 = 0, \\
& P(5)^2 P(3) P(1) - P(6) P(2)^2 P(4) + q^{13} P(6) P(1)^2 P(3) = 0, \\
& P(5)^2 P(4) P(2) - P(6) P(4) P(3)^2 + q^{26} P(5) P(2) P(1)^2 = 0, \\
& P(5)^2 P(6) P(1) - P(5) P(2) P(4)^2 + q^{13} P(4) P(1) P(3)^2 = 0, \\
& P(5)^2 P(6) P(2) - P(6) P(3) P(4)^2 + q^{26} P(4) P(1) P(2)^2 = 0, \\
& P(6)^2 P(3) P(1) - P(5) P(2)^2 P(6) + q^{13} P(4) P(1)^2 P(5) = 0, \\
& P(6)^2 P(4) P(1) - P(5)^2 P(2) P(3) + q^{13} P(4)^2 P(1) P(2) = 0, \\
& P(6)^2 P(4) P(2) - P(6) P(3)^2 P(5) + q^{26} P(4) P(1)^2 P(3) = 0, \\
& P(6)^2 P(5) P(1) - P(5) P(4) P(3)^2 + q^{13} P(4) P(3) P(2)^2 = 0, \\
& P(6)^2 P(5) P(2) - P(5)^2 P(4) P(3) + q^{26} P(3)^2 P(2) P(1) = 0, \\
& P(6)^2 P(5) P(3) - P(6) P(5) P(4)^2 + q^{39} P(3) P(2) P(1)^2 = 0, \\
& P(6) P(4) P(5) P(1) - P(6) P(3) P(4) P(2) + q^{13} P(5) P(2) P(3) P(1) = 0.
\end{aligned}$$

These equations all follow from (3.10). Doing so gives that the  $A'_{13,i}(q^{13})$  reduce to the  $A_{13,i}(q^{13})$  stated in Theorem 1.2. Given the number of identities coming from (3.10) and that we are working modulo 13, it may be possible to find  $A_{13,i}$  that consist of fewer terms. This finishes the proof of (1.7).  $\square$

*Proof of (1.8).* We recall here  $\ell = 13$  and  $P(a) = [q^{13a}; q^{169}]_\infty$ . We expand the  $\ell = 13$  case of (2.6) with Lemmas 2.1 and 2.2, using  $m = 1$  in each application, to obtain

$$\begin{aligned}
E(1)^3 V(q) \equiv & \frac{12q^{13} E(1)^3 T(91, 13, 169)}{E(169) P(1)} + \frac{6q^{13} E(169)^2 P(2)}{P(1) P(4)} + \frac{5q^{13} E(169)^2 P(5)}{P(3) P(4)} + \frac{2q^{13} E(169)^2 P(2) P(4)}{P(1) P(3) P(6)} + \frac{8q E(169)^2 P(3) P(6)}{P(1) P(2) P(5)} \\
& + \frac{12q E(169)^2 P(5)}{P(1) P(3)} + \frac{10q^{14} E(169)^2 P(4)}{P(2) P(6)} + \frac{4q^{14} E(169)^2 P(2) P(5)}{P(1) P(4) P(6)} + \frac{12q^2 E(169)^2 P(3)}{P(1) P(2)} + \frac{q^2 E(169)^2 P(2) P(5)}{P(1)^2 P(6)} \\
& + \frac{10q^{15} E(169)^2 P(2)}{P(1) P(5)} + \frac{4q^{15} E(169)^2}{P(3)} + \frac{11q^3 E(1)^3 T(39, 13, 169)}{E(169) P(1)} + \frac{11q^3 E(169)^2 P(4)}{P(1) P(3)} + \frac{4q^3 E(169)^2 P(6)}{P(1) P(4)} \\
& + \frac{12q^3 E(169)^2 P(3) P(5)}{P(1) P(2) P(6)} + \frac{2q^3 E(169)^2 P(2) P(4)}{P(1)^2 P(5)} + \frac{4q^{16} E(169)^2 P(2)}{P(1) P(6)} + \frac{4q^4 E(169)^2 P(4) P(5)}{P(1) P(3) P(6)} \\
& + \frac{3q^4 E(169)^2 P(2) P(3)}{P(1)^2 P(4)} + \frac{3q^4 E(169)^2 P(2) P(6)}{P(1) P(3)^2} + \frac{3q^4 E(169)^2 P(4)}{P(2)^2} + \frac{8q^4 E(169)^2 P(5)}{P(1) P(4)} + \frac{6q^{17} E(169)^2 P(6)}{P(4) P(5)} \\
& + \frac{10 E(169)^2 P(5)}{q^8 P(1)^2} + \frac{3 E(1)^3 T(13, 13, 169)}{q^8 E(169) P(1)} + \frac{4q^5 E(169)^2 P(2)^2}{P(1)^2 P(3)} + \frac{8q^5 E(169)^2 P(3) P(4)}{P(1) P(2) P(6)} + \frac{q^{18} E(169)^2 P(2) P(4)}{P(1) P(5) P(6)} \\
& + \frac{q^{18} E(169)^2}{P(4)} + \frac{9 E(169)^2 P(4) P(6)}{q^7 P(1)^2 P(5)} + \frac{7q^6 E(169)^2 P(5)}{P(2) P(3)} + \frac{8q^6 E(169)^2}{P(1)} + \frac{3q^6 E(169)^2 P(4)^2}{P(1) P(3) P(6)} \\
& + \frac{3q^{19} E(169)^2 P(2) P(3)}{P(1) P(4) P(6)} + \frac{12q^7 E(1)^3 T(52, 13, 169)}{E(169) P(1)} + \frac{4q^7 E(169)^2 P(3)^2}{P(1) P(2) P(5)} + \frac{7q^7 E(169)^2 P(3) P(5)}{P(1) P(4)^2} \\
& + \frac{7q^7 E(169)^2 P(2) P(6)}{P(1) P(3) P(4)} + \frac{5q^{20} E(169)^2}{P(5)} + \frac{11 E(169)^2 P(3) P(5)}{q^5 P(1)^2 P(4)} + \frac{12q^8 E(169)^2 P(4)}{P(1) P(5)} + \frac{11q^8 E(169)^2 P(6)}{P(2) P(4)} \\
& + \frac{4q^8 E(169)^2 P(4)}{P(2) P(3)} + \frac{11q^{21} E(169)^2}{P(6)} + \frac{12q^9 E(169)^2 P(4)}{P(1) P(6)} + \frac{7q^9 E(169)^2 P(2) P(6)}{P(1) P(3) P(5)} + \frac{8q^{22} E(169)^2 P(5)}{P(6)^2} \\
& + \frac{11q^{10} E(169)^2 P(2)}{P(1) P(3)} + \frac{2q^{10} E(169)^2 P(6)}{P(2) P(5)} + \frac{8q^{10} E(169)^2 P(3) P(5)}{P(1) P(4) P(6)} + \frac{2q^{23} E(169)^2 P(4)}{P(5) P(6)} + \frac{9 E(1)^3 T(26, 13, 169)}{q^2 E(169) P(1)} \\
& + \frac{7 E(169)^2 P(5)^2}{q^2 P(1) P(2) P(6)} + \frac{8 E(169)^2 P(2) P(4)}{q^2 P(1)^2 P(3)} + \frac{10 E(169)^2 P(4) P(6)}{q^2 P(1) P(2) P(5)} + \frac{10q^{11} E(169)^2 P(3)}{P(1) P(5)} + \frac{9q^{11} E(169)^2}{P(2)} \\
& + \frac{E(169)^2 P(3) P(6)}{q P(1) P(2) P(4)} + \frac{5q^{12} E(1)^3 T(78, 13, 169)}{E(169) P(1)} + \frac{6q^{12} E(169)^2 P(5)}{P(2) P(6)} + \frac{8q^{12} E(169)^2 P(2) P(4)}{P(1) P(3) P(5)}. \tag{3.12}
\end{aligned}$$

We multiply (3.12) by (3.11), collect terms, and reduce modulo 13, to find that

$$\begin{aligned}
V(q) \equiv & \frac{12q^{13} T(91, 13, 169)}{E(169) P(1)} + \frac{11q^3 T(39, 13, 169)}{E(169) P(1)} + \frac{3T(13, 13, 169)}{q^8 E(169) P(1)} + \frac{12q^7 T(52, 13, 169)}{E(169) P(1)} + \frac{9T(26, 13, 169)}{q^2 E(169) P(1)} + \frac{5q^{12} T(78, 13, 169)}{E(169) P(1)} \\
& + B'_{13}(q) \pmod{13},
\end{aligned}$$

where

$$\begin{aligned}
B'_{13}(q) &= \frac{E(169)^4}{E(13)} \left( B'_{13,0}(q^{13}) + qB'_{13,1}(q^{13}) + q^2B'_{13,2}(q^{13}) + q^3B'_{13,3}(q^{13}) + q^4B'_{13,4}(q^{13}) \right. \\
&\quad + q^5B'_{13,5}(q^{13}) + q^6B'_{13,6}(q^{13}) + q^7B'_{13,7}(q^{13}) + q^8B'_{13,8}(q^{13}) + q^9B'_{13,9}(q^{13}) + q^{10}B'_{13,10}(q^{13}) \\
&\quad \left. + q^{11}B'_{13,11}(q^{13}) + q^{12}B'_{13,12}(q^{13}) \right), \\
B'_{13,0}(q^{13}) &= \frac{11P(5)^2P(6)P(3)}{P(1)} + \frac{3P(6)^2P(3)^3}{P(1)P(2)} + \frac{P(5)^2P(4)^2}{P(1)} + \frac{12P(6)^3P(4)}{P(2)} + \frac{11P(5)^3P(6)}{P(2)} + \frac{7P(5)P(6)^2P(2)P(4)}{P(1)P(3)} \\
&\quad + \frac{7P(5)P(6)P(2)P(3)^2}{P(1)^2} + \frac{4q^{13}P(6)P(2)^3P(3)}{P(1)^2} + \frac{9q^{13}P(5)P(6)P(2)^2}{P(1)} + \frac{2q^{13}P(5)^2P(6)P(2)}{P(3)} \\
&\quad + \frac{7q^{13}P(5)P(3)^2P(4)}{P(2)} + \frac{q^{13}P(5)P(6)^2P(1)}{P(2)} + \frac{4q^{13}P(6)^2P(2)^3P(4)}{P(1)P(3)^2} + \frac{6q^{13}P(5)P(2)^2P(4)^2}{P(1)P(3)} \\
&\quad + \frac{7q^{13}P(5)^2P(2)P(3)^2}{P(1)P(4)} + \frac{6q^{13}P(5)P(3)P(4)^3}{P(6)P(2)} + \frac{11q^{13}P(5)^2P(2)P(3)P(4)}{P(6)P(1)} + \frac{4q^{13}P(5)P(6)P(1)P(4)^2}{P(2)P(3)} \\
&\quad + \frac{8q^{26}P(6)^2P(2)^2}{P(5)} + \frac{12q^{26}P(5)^2P(2)^2}{P(4)} + \frac{5q^{26}P(2)P(4)^3}{P(6)} + \frac{5q^{26}P(5)^3P(2)P(3)}{P(6)^2} + 6q^{26}P(2)P(3)P(4) \\
&\quad + 5q^{26}P(5)P(1)P(4) + \frac{q^{39}P(2)^2P(3)^2}{P(6)} + \frac{4q^{39}P(1)P(3)^3}{P(5)} + \frac{7q^{39}P(6)P(1)P(2)^2}{P(4)} + \frac{7q^{39}P(5)P(1)P(2)P(3)^2}{P(4)^2} \\
&\quad + \frac{5P(1)^2P(2)P(3)q^5}{P(5)^2}, \\
B'_{13,1}(q^{13}) &= \frac{12P(5)^2P(6)P(2)P(4)}{P(1)P(3)} + \frac{10P(6)P(4)^3}{P(1)} + \frac{11P(5)^3P(3)}{P(1)} + \frac{3P(5)P(6)P(2)^2P(4)}{P(1)^2} + \frac{9P(5)P(6)^3P(3)}{P(2)P(4)} \\
&\quad + \frac{7P(6)^2P(3)P(4)}{P(1)} + \frac{q^{13}P(6)P(2)^4P(4)}{P(1)^2P(3)} + \frac{8q^{13}P(6)^2P(2)^2P(4)}{P(5)P(1)} + \frac{12q^{13}P(2)P(3)P(4)^2}{P(1)} + \frac{4q^{13}P(6)^2P(2)P(4)}{P(3)} \\
&\quad + \frac{9q^{13}P(5)^2P(2)^2}{P(1)} + \frac{7q^{13}P(6)P(2)P(3)^2}{P(1)} + \frac{12q^{13}P(5)P(4)^4}{P(6)P(3)} + \frac{2q^{13}P(5)^2P(6)P(1)}{P(2)} + \frac{5q^{13}P(5)^3P(2)P(3)^2}{P(6)P(1)P(4)} \\
&\quad + \frac{2q^{13}P(5)^2P(1)P(4)^2}{P(2)P(3)} + \frac{3q^{13}P(5)^2P(3)^2P(4)}{P(6)P(2)} + \frac{3q^{13}P(1)P(6)P(3)P(4)^2}{P(2)^2} + 12q^{13}P(5)P(4)^2 \\
&\quad + 9q^{13}P(5)P(6)P(3) + \frac{9q^{26}P(5)P(2)P(3)P(4)}{P(6)} + \frac{12q^{26}P(1)P(5)P(6)P(2)}{P(3)} + 12q^{26}P(6)P(2)^2 \\
&\quad + \frac{8q^{26}P(5)^2P(1)P(4)}{P(6)} + \frac{6q^{26}P(6)^2P(1)P(3)}{P(5)} + \frac{10q^{26}P(3)^3P(4)}{P(5)} + \frac{10q^{26}P(2)^3P(4)^2}{P(5)P(1)} + \frac{9q^{26}P(2)P(5)P(3)^2}{P(4)} \\
&\quad + \frac{5q^{39}P(1)P(2)P(3)P(4)}{P(5)} + \frac{11q^{39}P(1)^2P(6)P(3)}{P(4)} + 4q^{39}P(1)^2P(4) + \frac{11q^{52}P(1)^2P(2)P(3)}{P(6)}, \\
B'_{13,2}(q^{13}) &= \frac{6P(5)^2P(2)^2P(4)}{P(1)^2} + \frac{6P(5)^4P(3)}{P(6)P(1)} + \frac{11P(6)^2P(3)^3P(4)}{P(5)P(1)P(2)} + \frac{9P(5)P(6)^2P(3)^2}{P(1)P(4)} + \frac{12P(6)^2P(2)^2P(4)^2}{P(5)P(1)^2} \\
&\quad + \frac{9P(5)P(6)P(3)P(4)}{P(1)} + \frac{12q^{13}P(5)P(6)P(2)P(4)}{P(3)} + \frac{6q^{13}P(5)^2P(6)^2P(1)}{P(3)P(4)} + \frac{4q^{13}P(1)P(5)P(3)P(4)^2}{P(2)^2} \\
&\quad + \frac{6q^{13}P(6)^2P(3)P(4)}{P(5)} + \frac{2q^{13}P(6)^2P(1)P(4)}{P(2)} + \frac{4q^{13}P(2)^2P(4)^3}{P(1)P(3)} + \frac{10q^{13}P(6)P(2)^2P(4)}{P(1)} + \frac{12q^{13}P(5)P(2)P(3)^2}{P(1)} \\
&\quad + \frac{q^{13}P(5)^2P(4)^2}{P(6)} + \frac{2q^{13}P(3)^2P(4)^2}{P(2)} + \frac{q^{13}P(5)P(6)^2P(2)}{P(4)} + 2q^{13}P(5)^2P(3) + \frac{4q^{26}P(2)^3P(3)}{P(1)} \\
&\quad + \frac{11q^{26}P(2)P(3)P(4)^2}{P(5)} + 12q^{26}P(5)P(2)^2 + 6q^{26}P(1)P(4)^2 + 4q^{26}P(1)P(3)P(6) + \frac{7q^{39}P(1)P(2)P(3)P(4)}{P(6)} \\
&\quad + \frac{7q^{39}P(6)P(1)P(2)^2}{P(5)} + \frac{8q^{52}P(5)P(1)^2P(2)P(3)}{P(6)^2}, \\
B'_{13,3}(q^{13}) &= \frac{4P(5)^2P(6)^2}{P(3)} + \frac{7P(5)^3P(4)}{P(2)} + \frac{7P(6)^3P(3)}{P(2)} + \frac{10P(6)^2P(4)^2}{P(2)} + \frac{4P(5)P(6)^2P(2)}{P(1)} + \frac{8P(5)^2P(3)P(4)}{P(1)} \\
&\quad + \frac{10P(6)P(3)^3P(4)}{P(1)P(2)} + \frac{6P(5)P(6)P(2)P(4)^2}{P(1)P(3)} + \frac{3P(5)P(2)P(3)^2P(4)}{P(1)^2} + \frac{12P(5)^2P(6)P(3)^2}{P(1)P(4)} \\
&\quad + \frac{9q^{13}P(5)P(6)^2P(2)^2}{P(3)^2} + \frac{12q^{13}P(5)^2P(2)P(4)}{P(3)} + \frac{7q^{13}P(5)^3P(3)}{P(6)} + \frac{3q^{13}P(5)P(6)P(2)^2P(3)}{P(1)P(4)} \\
&\quad + \frac{9q^{13}P(6)P(2)P(3)^2P(4)}{P(5)P(1)} + \frac{12q^{13}P(4)^3 + q^{13}3P(5)^2P(6)P(2)}{P(4)} + \frac{5q^{13}P(6)^2P(2)^3}{P(1)P(3)} + \frac{2q^{13}P(5)P(2)^2P(4)}{P(1)} \\
&\quad + \frac{3q^{13}P(1)P(5)P(4)^3}{P(2)P(3)} + \frac{2q^{13}P(5)P(3)^2P(4)^2}{P(6)P(2)} + \frac{3q^{13}P(1)P(5)P(6)P(4)}{P(2)} + 5q^{13}P(6)P(3)P(4) \\
&\quad + \frac{q^{26}P(2)P(3)P(4)^2}{P(6)} + \frac{5q^{26}P(6)^2P(1)P(2)}{P(4)} + \frac{5q^{26}P(5)P(1)P(4)^2}{P(6)} + \frac{9q^{26}P(6)P(2)^2P(4)}{P(5)} + 10q^{26}P(1)P(3)P(5) \\
&\quad + 3q^{26}P(2)P(3)^2 + \frac{2q^{39}P(6)P(1)^2P(3)}{P(5)} + \frac{7q^{39}P(5)P(1)P(2)P(3)P(4)}{P(6)^2} + \frac{8q^{39}P(5)P(1)P(2)P(3)^2}{P(6)P(4)} \\
&\quad + 11q^{39}P(1)P(2)^2 + \frac{2q^{52}P(1)^2P(2)P(3)P(4)}{P(5)P(6)},
\end{aligned}$$

$$\begin{aligned}
B'_{13,4}(q^{13}) = & \frac{3P(5)P(6)^2P(2)^2P(4)}{P(1)P(3)^2} + \frac{6P(6)P(2)P(3)^2P(4)^2}{P(5)P(1)^2} + \frac{9P(5)P(6)P(2)^2P(3)}{P(1)^2} + \frac{4P(5)^2P(2)P(4)^2}{P(1)P(3)} + \frac{9P(6)^2P(3)^2}{P(1)} \\
& + \frac{10P(5)P(3)^3P(4)}{P(1)P(2)} + \frac{12P(5)^2P(6)P(2)}{P(1)} + \frac{6P(6)P(3)P(4)^2}{P(1)} + \frac{3P(5)P(6)P(4)^2}{P(2)} + \frac{5P(5)P(6)^2P(3)}{P(2)} \\
& + \frac{12q^{13}P(5)P(6)P(2)^3}{P(1)P(3)} + \frac{q^{13}P(6)P(2)^2P(4)^2}{P(5)P(1)} + \frac{11q^{13}P(5)^2P(2)^2P(4)}{P(6)P(1)} + \frac{4q^{13}P(5)^2P(2)^2P(3)}{P(1)P(4)} \\
& + \frac{4q^{13}P(6)P(3)^3P(4)}{P(5)P(2)} + \frac{11q^{13}P(2)P(6)P(4)^2}{P(3)} + \frac{8q^{13}P(5)^3P(2)}{P(4)} + \frac{7q^{13}P(5)P(6)P(3)^2}{P(4)} + \frac{11q^{13}P(5)^2P(1)P(4)}{P(2)} \\
& + \frac{12q^{13}P(2)P(3)^2P(4)}{P(1)} + \frac{10q^{13}P(5)P(6)^2P(1)}{P(3)} + 4q^{13}P(5)P(3)P(4) + 5q^{13}P(6)^2P(2) + \frac{3q^{26}P(1)P(5)P(6)P(2)}{P(4)} \\
& + \frac{7q^{26}P(6)P(1)P(3)P(4)}{P(5)} + \frac{7q^{26}P(5)^2P(1)P(3)}{P(6)} + \frac{11q^{26}P(5)P(2)P(3)^2}{P(6)} + \frac{6q^{26}P(5)^2P(1)P(4)^2}{P(6)^2} \\
& + 12q^{26}P(2)^2P(4) + \frac{10q^{39}P(1)P(2)P(3)^2}{P(5)} + \frac{5q^{39}P(1)P(2)P(3)P(4)^2}{P(5)P(6)} + 9q^{39}P(1)^2P(3), \\
B'_{13,5}(q^{13}) = & \frac{10P(5)^2P(6)P(2)P(4)}{q^{13}P(1)^2} + \frac{10P(5)P(6)^3}{P(4)} + \frac{P(5)^3P(2)}{P(1)} + \frac{8P(5)P(6)^2P(4)}{P(3)} + \frac{5P(5)P(6)P(3)^2}{P(1)} \\
& + \frac{4P(5)^2P(6)P(3)}{P(2)} + \frac{9P(6)P(3)^2P(4)^2}{P(2)^2} + \frac{9P(5)^2P(2)^2P(3)}{P(1)^2} + \frac{2P(6)^2P(2)P(4)}{P(1)} + \frac{9P(6)P(2)P(3)^3}{P(1)^2} \\
& + \frac{7P(5)P(3)P(4)^2}{P(1)} + \frac{4P(5)P(6)P(2)^3P(4)}{P(1)^2P(3)} + \frac{5q^{13}P(6)^2P(2)^3P(4)}{P(5)P(1)P(3)} + \frac{9q^{13}P(5)^3P(2)^2P(3)}{P(6)P(1)P(4)} + \frac{q^{13}P(3)^3P(4)}{P(2)} \\
& + \frac{11q^{13}P(6)^2P(1)P(3)}{P(2)} + \frac{11q^{13}P(6)^2P(1)P(3)}{P(2)} + \frac{4q^{13}P(2)^2P(4)^2}{P(1)} + \frac{3q^{13}P(5)P(2)P(4)^2}{P(3)} \\
& + \frac{11q^{13}P(6)P(3)P(4)^2}{P(5)} + \frac{7q^{13}P(5)^2P(3)P(4)}{P(6)} + \frac{12q^{13}P(6)P(1)P(4)^2}{P(2)} + \frac{5q^{13}P(6)^2P(3)^2}{P(5)} + \frac{5q^{13}P(5)^2P(3)^2}{P(4)} \\
& + \frac{4q^{13}P(6)P(2)^2P(3)}{P(1)} + 6q^{13}P(5)P(6)P(2) + \frac{8q^{26}P(1)P(4)^3}{P(6)} + \frac{2q^{26}P(5)P(2)^2P(4)}{P(6)} + \frac{9q^{26}P(5)P(2)^2P(3)}{P(4)} \\
& + \frac{12q^{26}P(2)P(3)^2P(4)}{P(5)} + \frac{q^{26}P(6)P(1)P(3)^2}{P(4)} + 12q^{26}P(1)P(3)P(4) + \frac{8q^{39}P(1)P(2)^2P(4)}{P(5)} \\
& + \frac{6q^{39}P(5)P(1)^2P(3)}{P(6)}, \\
B'_{13,6}(q^{13}) = & \frac{4P(5)P(6)P(3)^2P(4)}{q^{13}P(1)^2} + \frac{9P(6)^2P(2)P(4)^2}{q^{13}P(1)^2} + \frac{11P(5)P(6)^2P(2)P(3)}{P(1)P(4)} + \frac{11P(5)^2P(3)^2}{P(1)} + \frac{P(6)P(4)^3}{P(2)} \\
& + \frac{7P(5)^2P(6)^2}{P(4)} + \frac{P(5)P(6)^3P(2)}{P(3)^2} + \frac{3P(5)^2P(6)P(4)}{P(3)} + \frac{2P(5)^3P(4)^2}{P(6)P(2)} + \frac{11P(3)^3P(4)^2}{P(1)P(2)} + \frac{8P(6)^2P(3)P(4)}{P(2)} \\
& + \frac{P(5)P(6)^2P(1)P(4)}{P(2)^2} + \frac{4P(6)P(2)^2P(3)P(4)}{P(1)^2} + \frac{12P(5)^3P(2)^2P(3)}{P(6)P(1)^2} + \frac{9P(5)P(2)P(4)^3}{P(1)P(3)} + \frac{11P(6)^2P(3)^2P(4)}{P(5)P(1)} \\
& + \frac{6q^{13}P(6)P(2)^3P(4)}{P(1)P(3)} + \frac{8q^{13}P(5)P(6)^2P(2)^2}{P(3)P(4)} + \frac{3q^{13}P(2)P(3)^2P(4)^2}{P(5)P(1)} + \frac{8q^{13}P(5)^2P(6)P(2)P(3)}{P(4)^2} \\
& + \frac{6q^{13}P(5)P(6)P(1)P(3)}{P(2)} + \frac{2q^{13}P(6)^3P(1)}{P(4)} + \frac{q^{13}P(6)^2P(2)P(4)}{P(5)} + \frac{8q^{13}P(5)P(1)P(4)^2}{P(2)} + \frac{4q^{13}P(5)P(2)^2P(3)}{P(1)} \\
& + 11q^{13}P(6)P(3)^2 + 7q^{13}P(3)P(4)^2 + \frac{3q^{26}P(5)P(1)P(3)P(4)}{P(6)} + \frac{q^{26}P(2)^2P(4)^2}{P(5)} + 2q^{26}P(6)P(1)P(2) \\
& + \frac{5q^{39}P(5)P(1)^2P(2)}{P(4)} + \frac{2q^{39}P(1)P(2)^2P(4)}{P(6)} + \frac{6q^{39}P(1)P(2)^2P(3)}{P(4)}, \\
B'_{13,7}(q^{13}) = & \frac{12P(5)^2P(6)^2}{q^{13}P(1)} + \frac{P(6)^2P(3)^2P(4)^2}{q^{13}P(5)P(1)^2} + \frac{9P(3)P(4)^3}{P(1)} + \frac{5P(5)^2P(2)P(4)}{P(1)} + \frac{2P(6)P(2)^3P(4)^2}{P(1)^2P(3)} \\
& + \frac{P(5)P(2)^2P(3)P(4)}{P(1)^2} + \frac{12P(6)P(3)^2P(4)}{P(1)} + \frac{P(5)^3P(3)^2}{P(6)P(1)} + \frac{5P(5)P(6)^2P(2)^2}{P(1)P(3)} + \frac{9P(6)^2P(2)P(4)^2}{P(5)P(1)} \\
& + \frac{6P(5)P(6)P(3)P(4)}{P(2)} + \frac{P(5)P(6)P(2)^2P(3)^2}{P(1)^2P(4)} + \frac{4P(5)^2P(6)P(2)P(3)}{P(1)P(4)} + \frac{6q^{13}P(2)P(4)^3}{P(3)} + \frac{9q^{13}P(5)^2P(2)^2P(3)}{P(6)P(1)} \\
& + \frac{3q^{13}P(5)P(6)P(1)P(4)}{P(3)} + \frac{11q^{13}P(5)^2P(1)P(4)^2}{P(6)P(2)} + \frac{9q^{13}P(2)P(3)^3}{P(1)} + \frac{2q^{13}P(6)^2P(2)P(3)}{P(4)} \\
& + \frac{3q^{13}P(5)P(3)P(4)^2}{P(6)} + \frac{3q^{13}P(5)P(6)^2P(1)}{P(4)} + \frac{2q^{13}P(6)^2P(1)P(3)P(4)}{P(5)P(2)} + 8q^{13}P(5)P(3)^2 + 9q^{13}P(6)P(2)P(4) \\
& + \frac{2q^{26}P(5)^2P(1)P(3)P(4)}{P(6)^2} + \frac{2q^{26}P(1)P(3)P(4)^2}{P(5)} + \frac{8q^{26}P(6)P(1)P(3)^2}{P(5)} + \frac{5q^{26}P(2)^2P(4)^2}{P(6)} + 2q^{26}P(2)^2P(3) \\
& + 12q^{26}P(5)P(1)P(2) + \frac{10q^{39}P(1)^2P(3)P(4)}{P(6)} + \frac{4q^{39}P(5)P(1)P(2)^2P(3)}{P(6)P(4)}, \\
B'_{13,8}(q^{13}) = & \frac{3P(6)^3P(4)}{q^{13}P(1)} + \frac{10P(5)^2P(6)P(2)P(3)}{q^{13}P(1)^2} + \frac{6P(6)P(2)^2P(3)P(4)^2}{P(5)P(1)^2} + \frac{12P(6)P(3)^4P(4)}{P(5)P(1)P(2)} + \frac{10P(5)^2P(6)P(2)^2}{P(1)P(3)} \\
& + \frac{10P(5)^2P(2)^2P(3)^2}{P(1)^2P(4)} + \frac{9P(5)^3P(2)P(4)}{P(6)P(1)} + \frac{3P(5)^2P(3)P(4)}{P(2)} + \frac{10P(5)P(6)P(2)^3}{P(1)^2} + \frac{3P(6)P(2)P(4)^2}{P(1)} \\
& + \frac{4P(6)^2P(2)P(3)}{P(1)} + \frac{3P(5)P(6)P(4)^2}{P(3)} + \frac{4P(5)P(3)^2P(4)}{P(1)} + \frac{5P(5)^3P(2)P(3)}{P(1)P(4)} + \frac{8P(5)P(6)P(3)^3}{P(1)P(4)}
\end{aligned}$$



$$\begin{aligned}
& + \frac{10P(6)^2P(3)P(4)^2}{P(5)P(2)} + \frac{11P(5)^2P(6)^2P(1)}{P(2)P(3)} + 9P(5)P(6)^2 + \frac{8q^{13}P(5)P(2)P(4)^3}{P(6)P(3)} + \frac{4q^{13}P(5)P(6)P(2)P(3)}{P(4)} \\
& + \frac{3q^{13}P(6)P(1)P(3)P(4)}{P(2)} + \frac{2q^{13}P(6)P(2)^3P(4)^2}{P(5)P(1)P(3)} + \frac{8q^{13}P(6)^2P(2)^2}{P(3)} + \frac{2q^{13}P(5)^2P(3)^2}{P(6)} + \frac{6q^{13}P(6)P(3)^2P(4)}{P(5)} \\
& + \frac{8q^{13}P(2)^2P(3)P(4)}{P(1)} + \frac{7q^{13}P(5)^2P(1)P(4)}{P(3)} + 2q^{13}P(5)P(2)P(4) + \frac{11q^{26}P(5)^2P(1)P(2)}{P(6)} + \frac{6q^{26}P(1)P(3)P(4)^2}{P(6)} \\
& + \frac{11q^{26}P(5)P(2)^2P(3)}{P(6)} + 12q^{26}P(1)P(3)^2 + \frac{10q^{39}P(1)P(2)^2P(3)}{P(5)} + 4q^{39}P(1)^2P(2), \\
B'_{13,9}(q^{13}) = & \frac{3P(6)^2P(2)P(3)P(4)}{q^{13}P(1)^2} + \frac{3P(5)^3P(2)P(3)}{q^{13}P(1)^2} + \frac{7P(5)P(6)P(3)^3}{q^{13}P(1)^2} + \frac{11P(5)P(6)^3P(2)}{P(3)P(4)} + \frac{10P(6)P(3)^2P(4)^2}{P(5)P(1)} \\
& + \frac{3P(5)P(6)^2P(2)P(4)}{P(3)^2} + \frac{8P(5)P(6)P(2)P(3)}{P(1)} + \frac{9P(5)^2P(2)^3}{P(1)^2} + \frac{5P(6)^2P(3)^2}{P(2)} + \frac{11P(5)^2P(6)^2P(3)}{P(4)^2} \\
& + \frac{10P(6)P(3)P(4)^2}{P(2)} + \frac{9P(6)P(2)^2P(3)^2}{P(1)^2} + \frac{P(5)P(2)P(4)^2}{P(1)} + \frac{5P(5)^3P(3)P(4)}{P(6)P(2)} + \frac{3P(5)P(6)P(1)P(4)^2}{P(2)^2} \\
& + \frac{5P(5)^2P(3)^2P(4)}{P(6)P(1)} + \frac{3P(6)^2P(2)^2P(4)}{P(1)P(3)} + 6P(5)^2P(6) + \frac{q^{13}P(5)P(2)^2P(3)^2}{P(1)P(4)} + \frac{q^{13}P(5)P(1)P(4)^3}{P(6)P(2)} \\
& + \frac{7q^{13}P(2)^3P(4)^2}{P(1)P(3)} + \frac{5q^{13}P(5)P(6)P(2)^2}{P(3)} + \frac{8q^{13}P(6)P(2)P(4)^2}{P(5)} + \frac{11q^{13}P(5)^2P(2)P(4)}{P(6)} + \frac{5q^{13}P(6)^2P(2)P(3)}{P(5)} \\
& + \frac{10q^{13}P(5)^2P(2)P(3)}{P(4)} + \frac{8q^{13}P(5)P(2)^2P(3)P(4)}{P(6)P(1)} + \frac{8q^{13}P(6)P(2)^3}{P(1)} + \frac{5q^{13}P(5)P(1)P(3)P(4)}{P(2)} + 7q^{13}P(3)^2P(4) \\
& + 5q^{13}P(6)^2P(1) + \frac{q^{26}P(2)^2P(3)P(4)}{P(5)} + \frac{12q^{26}P(5)P(1)P(3)^2}{P(6)} + \frac{4q^{26}P(6)P(1)P(2)P(3)}{P(4)} + q^{26}P(1)P(2)P(4) \\
& + \frac{4q^{39}P(1)P(2)^2P(3)}{P(6)}, \\
B'_{13,10}(q^{13}) = & \frac{10P(5)^2P(6)P(4)}{q^{13}P(1)} + \frac{8P(5)P(6)P(2)P(3)P(4)}{q^{13}P(1)^2} + \frac{8P(5)^2P(6)^2P(3)}{q^{13}P(1)P(4)} + \frac{3P(5)P(6)P(2)^2P(4)}{P(1)P(3)} \\
& + \frac{10P(6)^2P(2)P(3)P(4)}{P(5)P(1)} + \frac{11P(5)^2P(6)P(2)P(3)^2}{P(1)P(4)^2} + \frac{10P(5)P(6)^2P(1)P(4)}{P(2)P(3)} + \frac{11P(5)P(6)^2P(2)^2}{P(1)P(4)} \\
& + \frac{P(5)^2P(2)P(4)^2}{P(6)P(1)} + \frac{8P(5)P(6)^3P(1)}{P(2)P(4)} + \frac{12P(6)P(2)^3P(4)}{P(1)^2} + \frac{4P(5)P(3)P(4)^2}{P(2)} + \frac{10P(5)P(6)P(3)^2}{P(2)} \\
& + \frac{8P(3)^2P(4)^2}{P(1)} + \frac{10P(6)P(3)^3}{P(1)} + P(5)^3 + 10P(6)^2P(4) + \frac{3q^{13}P(5)P(1)P(4)^2}{P(3)} + \frac{2q^{13}P(5)P(3)^2P(4)}{P(6)} \\
& + \frac{q^{13}P(5)P(2)^3}{P(1)} + \frac{10q^{13}P(5)^2P(2)^2P(3)^2}{P(6)P(1)P(4)} + \frac{8q^{13}P(5)^2P(1)P(3)P(4)}{P(6)P(2)} + \frac{8q^{13}P(2)^2P(3)P(4)^2}{P(5)P(1)} \\
& + 3q^{13}P(6)P(2)P(3) + 2q^{13}P(5)P(6)P(1) + 12q^{13}P(2)P(4)^2 + \frac{10q^{26}P(2)^2P(3)P(4)}{P(6)} + \frac{3q^{26}P(6)P(1)P(2)^2}{P(3)} \\
& + \frac{3q^{26}P(1)^2P(3)P(4)}{P(2)} + \frac{4q^{26}P(5)P(1)P(2)P(4)}{P(6)} + \frac{8q^{26}P(5)P(1)P(2)P(3)}{P(4)} + \frac{3q^{26}P(2)^2P(3)^2}{P(4)} \\
& + \frac{6q^{39}P(6)P(1)^2P(2)P(3)}{P(5)P(4)}, \\
B'_{13,11}(q^{13}) = & \frac{6P(6)^2P(4)^2}{q^{13}P(1)} + \frac{7P(5)^3P(4)}{q^{13}P(1)} + \frac{8P(5)P(6)P(2)^2P(4)^2}{q^{13}P(1)^2P(3)} + \frac{P(6)^2P(2)P(3)P(4)^2}{q^{13}P(5)P(1)^2} + \frac{8P(5)^2P(6)P(2)P(3)^2}{q^{13}P(1)^2P(4)} \\
& + \frac{7P(5)^2P(6)P(1)P(4)}{P(2)P(3)} + \frac{6P(5)^2P(6)P(2)^2}{P(1)P(4)} + \frac{9P(6)P(2)P(3)P(4)}{P(1)} + \frac{6P(5)^3P(2)P(3)^2}{P(1)P(4)^2} + \frac{9P(5)^2P(3)P(4)^2}{P(6)P(2)} \\
& + \frac{P(6)^2P(3)^2P(4)}{P(5)P(2)} + \frac{10P(5)P(2)^3P(4)}{P(1)^2} + \frac{11P(5)P(6)^2P(3)}{P(4)} + \frac{4P(2)P(4)^3}{P(1)} + \frac{11P(6)^3P(2)}{P(3)} + \frac{8P(5)P(4)^3}{P(3)} \\
& + \frac{7P(5)P(3)^3}{P(1)} + 2P(5)P(6)P(4) + \frac{7q^{13}P(5)P(2)P(4)^2}{P(6)} + \frac{11q^{13}P(1)P(3)P(4)^2}{P(2)} + \frac{10q^{13}P(2)^2P(3)^2}{P(1)} \\
& + \frac{6q^{13}P(3)^2P(4)^2}{P(5)} + \frac{9q^{13}P(6)P(2)^3P(4)}{P(5)P(1)} + 9q^{13}P(5)P(2)P(3) + 10q^{13}P(5)^2P(1) + \frac{8q^{26}P(1)P(3)^2P(4)}{P(6)} \\
& + \frac{2q^{26}P(6)P(1)P(2)P(3)}{P(5)} + 4q^{26}P(2)^3 + \frac{q^{39}P(1)^2P(2)P(3)}{P(4)} + \frac{q^{39}P(1)P(2)^2P(3)P(4)}{P(5)P(6)}, \\
B'_{13,12}(q^{13}) = & \frac{P(5)P(6)^2P(3)}{q^{13}P(1)} + \frac{4P(6)^2P(3)^2P(4)^2}{q^{13}P(5)P(1)P(2)} + \frac{11P(6)P(2)P(3)P(4)^2}{q^{13}P(1)^2} + \frac{2P(5)^3P(2)P(3)^2}{q^{13}P(1)^2P(4)} + \frac{8P(5)^2P(3)^2P(4)}{q^{13}P(1)P(2)} \\
& + \frac{8P(5)P(6)P(2)P(3)^2}{P(1)P(4)} + \frac{7P(6)P(2)^3P(4)^2}{P(5)P(1)^2} + \frac{P(6)P(2)^2P(4)^2}{P(1)P(3)} + \frac{12P(5)P(6)P(2)P(4)^2}{P(3)^2} + \frac{7P(5)P(2)P(3)P(4)}{P(1)} \\
& + \frac{3P(6)P(3)^3P(4)}{P(5)P(1)} + \frac{2P(5)P(6)^2P(2)}{P(3)} + \frac{3P(5)^2P(4)^3}{P(6)P(3)} + \frac{3P(5)^2P(6)P(3)}{P(4)} + \frac{6P(6)P(3)^2P(4)}{P(2)} \\
& + \frac{12P(5)P(1)P(4)^3}{P(2)^2} + \frac{5P(6)^3P(1)}{P(2)} + \frac{7P(6)^2P(2)^2}{P(1)} + 11P(5)^2P(4) + \frac{6q^{13}P(5)P(2)^2P(4)}{P(3)} + \frac{5q^{13}P(5)P(6)P(2)^2}{P(4)} \\
& + \frac{10q^{13}P(5)^2P(2)P(3)}{P(6)} + \frac{11q^{13}P(2)^3P(4)}{P(1)} + \frac{7q^{13}P(3)^2P(4)^2}{P(6)} + \frac{2q^{13}P(5)^2P(6)P(1)P(2)}{P(3)P(4)} \\
& + \frac{9q^{13}P(5)P(1)P(3)P(4)^2}{P(6)P(2)} + \frac{8q^{13}P(6)P(2)P(3)P(4)}{P(5)} + 12q^{13}P(6)P(1)P(4) + \frac{3q^{26}P(1)P(2)P(4)^2}{P(6)}
\end{aligned}$$

$$+ \frac{9q^{26}P(2)^2P(3)P(4)^2}{P(5)P(6)} + 4q^{26}P(1)P(2)P(3) + 7q^{26}P(5)P(1)^2 + \frac{3q^{39}P(1)P(2)^2P(3)^2}{P(6)P(4)}.$$

To finish the proof of (1.8) we must show that  $B'_{13} \equiv B_{13} \pmod{13}$ . We do this by reducing each  $B'_{13,i}$  modulo 13 in the same way that we reduced the  $A'_{13,i}$  in the proof of (1.7). Doing so gives that the  $B'_{13,i}$  reduce modulo 13 to the  $B_{13,i}$  defined in Theorem 1.2.  $\square$

#### 4. REMARKS

These functions satisfy additional congruences. For example, it appears that

$$\begin{aligned} u(9n) &\equiv 0 \pmod{9}, \\ v(9n+1) &\equiv 0 \pmod{9}, \\ v(27n+1) &\equiv 0 \pmod{27}. \end{aligned}$$

One could use the methods of this article to prove these congruences, however the difficulty in using these methods when  $\ell$  is not a prime is that we can not as easily reduce  $E(1)^{-3}$  modulo  $\ell$ . In particular, with  $\ell = 9$ , it is not the case that  $\frac{1}{E(1)^3} \equiv \frac{E(1)^6}{E(9)^3}$ , but rather we instead have from  $\frac{E(1)^9}{E(3)^3} \equiv 1 \pmod{9}$  that

$$\frac{1}{E(1)^3} \equiv \frac{E(1)^6 E(3)^6}{E(9)^3} \pmod{9}.$$

One would first have to find a short representation of the 9-dissection taken modulo 9, or simply deal with the bothersome number of terms introduced by expanding this with (2.4). One could also verify the required terms are zero modulo  $\ell$  by viewing the eta quotients and generalized eta quotients as modular forms, however that is also rather computational proof.

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